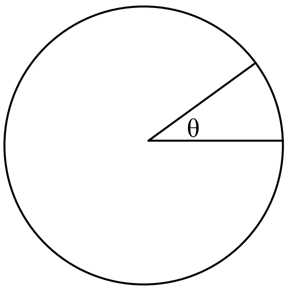
Maths Revision Notes

Functions and Graphs

Circles



Area of Circle: $A = \pi r^2$

Area of sector $= \dfrac{\theta}{2\pi}\pi r^2 = \dfrac{\theta}{2} r^2$

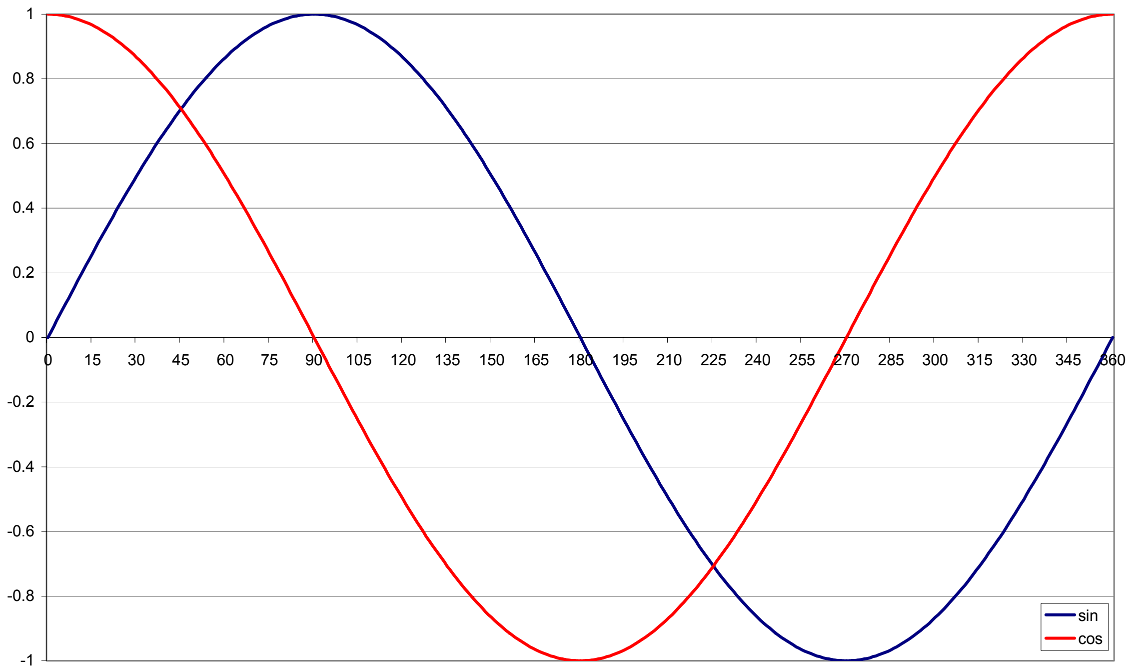
Circumference $= 2\pi r$

Arc Length $\dfrac{\theta}{2\pi} 2\pi r = \theta r$

The equation of a circle: $(x-a)^2 + (y-b)^2 = r^2$
a is the *x* offset, *b* is the *y* offset.

Trigonometry

Remember what the graphs look like:



So general solutions are:

Remember radians. 2π in a circle. Degrees → radians: × π ÷ 180.
So: 180° = π, 90° = π/2, 30° = π/6

Remember: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Also remember: $\sec\theta = \dfrac{1}{\cos\theta}$ Remember that and the rest (cosec = sin, cot = tan) are easy.

Logarithms
Just remember the rules.

$\log_a b = c \Leftrightarrow a^c = b$

$\therefore \ln b = c \Leftrightarrow e^c = b$

$\log_a c = \dfrac{\log_b c}{\log_b a} = \dfrac{\ln c}{\ln a}$

$\log bc = \log b + \log c$

$\log\left(\dfrac{b}{c}\right) = \log b - \log c$

$\log b^n = n \log b \;\Rightarrow\; \log \tfrac{1}{b} = \log b^{-1} = -\log b$

Remember with log-log graphs, line is equivalent to $y = mx + c$, i.e.

$y = ax^n \Rightarrow \log y = \log a + n \log x$

Straight Line Graphs
Equation for straight line is: $y = mx + c$

Gradient can be calculated as: $\dfrac{y_2 - y_1}{x_2 - x_1} = m$

If you know a point (x_2 and y_2) and gradient (m), can find y-intercept: y_1 where x_1=0.

For parallel lines, gradients are the same. For perpendicular lines: $m_1 = -\dfrac{1}{m_2} \therefore m_1 m_2 = -1$

Functions
'Even' when $f(x) = f(-x)$, i.e. when symmetrical around *y* axis (e.g. cos).
'Odd' when $f(x) = -f(-x)$, i.e. when rotated 180° around the origin, i.e. reflected in both *x* and *y* axes (e.g. sin).
A periodic function is one with a repeating pattern (e.g. sin/cos/tan).
Inverse function $f^{-1}(x)$ is like swapping *x* and *y* in *y=x*, alternatively expressed as a reflection of f(*x*) in the line y=x. Only has an inverse if would have single answer (so not one for √).
Other things to note: $gf(x) = g(f(x)) \Rightarrow f^2(x) = f(f(x))$

Transformations
$y = f(x) + c$ shifts curve up *y* axis by *c*
$y = f(x + c)$ shifts curve leftwards along *x* axis by *c*
$y = -f(x)$ reflection of curve in *x* axis
$y = f(-x)$ reflection of curve in *y* axis
$y = af(x)$ stretches curve up *y* axis by factor *a*
$y = f(ax)$ stretches curve along *x* axis by factor 1/*a*, i.e. shrinks curve

Inverse Trig Functions
These too reflect in *y*=*x*, but only within a domain of –1 to 1, as the pattern then repeats.
Important trick: substitute inverse trig functions when compound, e.g.
$\sin(2\cos^{-1} x) \quad \cos^{-1} x = \alpha \quad \therefore \sin 2\alpha = 2 \sin\alpha \cos\alpha \quad \cos\alpha = x$ *etc.*

rsinθ
$r\sin(\theta + \alpha) \equiv a\cos\theta + b\sin\theta$

$\therefore r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \equiv a\cos\theta + b\sin\theta$

$r\sin\alpha = a \qquad r\cos\alpha = b$

$r = \sqrt{a^2 + b^2} \qquad \dfrac{r\sin\alpha}{r\cos\alpha} = \dfrac{a}{b} \quad \therefore \quad \tan\alpha = \dfrac{a}{b}$

Modulus of a Function

The output must be positive, in other words (and this is useful for locating intersections):

$$| f(x) | = f(x) \qquad f(x) \geq 0$$
$$= -f(x) \qquad f(x) < 0$$

Approximations

The *Interval Bisection Method* is basically:

- Locate two points around a root (e.g. *x*=1, *x*=2)
- Put into equation and see what answers are (e.g. f(1) < 0, f(2) > 0)
- So point lies between them. One iteration is to halve that and see what that gives (e.g. f(1.5) < 0, so point lies between 1.5 and 2).
- Halve again as a second iteration.
- Continue for as long as you want. Answer is closest approximation put into f(*x*).

The *Newton-Raphson* method is easy. Once you have a rough approximate point (*a*), a better approximation (*b*) is given by:

$$b = a \frac{f(a)}{f'(a)}$$

Local Linear Approximations

For small values of *h* (horizontal offset), the following will be an approximation:

$$f(a+h) \approx f(a) + hf'(a)$$

When asked for just f(*h*), remember *a* can be made zero.

Derivation is from:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

Where *h* is a short distance along from point *a*. Confused? All it's doing is finding a point a little way along the curve and working out the average gradient between that and the original point (*a*), which is near enough the gradient at *a*. The average gradient is δ*y*÷δ*x*, and this is visible in the above formula, as the difference between the two f's is the δ*y*, and the δ*x* is the movement along the *x* axis, which is *h*.

Parametric Equations

Where *x* and *y* are in terms of a third factor, e.g. *t* (time). To convert a parametric equation to Cartesian form, e.g. for drawing, write *t* in terms of *x*, and place in *y*=, e.g.:

$$x = 30t \qquad y = 2 - 5t^2 \qquad t = \tfrac{x}{30} \quad \therefore\ y = 2 - 5(\tfrac{x}{30})^2$$

Other Graph and Function Related Things

Domain: The set of inputs for a function (*x* axis).

Range: The set out outputs for a function (*y* axis).

Note that an inverse function will swap the domain and the range.

Domain of composite function fg(*x*) is the set of all *x* in the domain of g for which g(*x*) is the domain of f.

Inequalities

- Sign unchanged when number added/subtracted to both sides.
- Sign unchanged when multiplied/divided by *positive* number on both sides.
- Sign *changes* when multiplied/divided by negative number on both sides.

Not everything is that easy. With fractions with unknown denominators, you do not know whether they are positive or negative, so cannot multiply directly. Instead, multiply by denominator *squared* (cannot be negative), e.g.

$$\frac{x-2}{x-5} > 3 \qquad \times (x-5)^2 \qquad \Rightarrow (x-2)(x-5) > 3(x-5)^2$$

Then, subtract RHS from LHS – you've got a quadratic = 0, so can be solved.
More good examples in Gold Book, CH13.

Algebra Stuff

- Distinction between equation and identity (apparently).
- Solving quadratics (any method can be used, so use quadratic equation).
- Solving simultaneous equations with two unknowns (including quadratics):
  - Get one into the form y= and replace y in the other equation with it, to get x=
  - Same with quadratics. Arrange equations into a quadratic equation form and use the formula.
- Binomial Theorem (see formula book – 'power series').

Rationalising a denominator:
When a denominator is in surd form, just multiply whole fraction by denominator:

$$\frac{2}{\sqrt{3}} \Rightarrow \frac{2\sqrt{3}}{3}$$

Special case:

$$(a-\sqrt{b})(a+\sqrt{b}) = a^2 - b$$

$$\frac{3\sqrt{2}}{5-\sqrt{2}} \times (5+\sqrt{2}) = \frac{15\sqrt{2}+6}{23}$$

Remainder Theorem
If a polynomial is divided by a linear factor $(x - a)$ the remainder will be $f(a)$.
Therefore $f(a) = 0$ if $(x - a)$ is a factor. That is method for testing what factor to divide by.

Binomial Expansion
i.e. expanding $(x + y)^n$. There is the full equation ('power series') in the formula book for complicated expansions (mostly needed when you have a negative power), but Pascal's triangle can be used for simpler, positive power expansions:

1 1	n = 1
1 2 1	n = 2
1 3 3 1	n = 3
1 4 6 4 1	n = 4
1 5 10 10 5 1	n = 5

This is applied as follows:

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The pattern is visible. Each term has as its coefficient the number from Pascal's triangle. The *x* starts off with the *n* power, while the *y* starts with a power of zero (i.e. one, so it makes no difference). The powers of *x* decrease, while those of *y* increase.

Indices

$x^a \times x^b = x^{a+b}$ $x^a \div x^b = x^{a-b}$ $(x^a)^b = x^{ab}$ $x^{-a} = \frac{1}{x^a}$ $x^{\frac{1}{a}} = \sqrt[a]{x}$ $x^{\frac{a}{b}} = \sqrt[b]{x^a}$ $x^0 = 1$

Sequences and Series (AP's and GP's)

A sequence cab be classified by how it behaves as it moves towards infinity:
Convergent: if the values tend towards a finite value
Divergent: if they don't (i.e. they get further and further away, e.g. exponential).
Oscillating: if the values does not converge or diverge from a point. If regular pattern, is said to be periodic function (see earlier).

A series is the sum of the values in a sequence.

Basic use of sigma. An example of the sigma (sum) notation is as follows:

$$\sum_{r=2}^{10} r^3 = 2^3 + 3^3 + \ldots + 10^3$$

Arithmetic Progressions (AP's)
An AP increments each term by a set amount, e.g. 5, 8, 11, 14 – so the starting number, *a*, is 5, and the difference, *d*, is 3. An AP with *n* terms can be written as:

$$a, (a+d), (a+2d), (a+3d), \ldots, (a+(n-1)d)$$

The final expression, $a+(n-1)d$, is the general expression for the n^{th} term.
The sum of *n* terms of an AP can be expressed as:

$$S_n = \tfrac{1}{2}n(a+l)$$

Where: *n* = number of terms, *a* = starting term, *l* = last term.
If the final term is unknown, the earlier expression can be combined with the above equation to give this alternative expression for a sum of an AP:

$$S_n = \tfrac{1}{2}n(2a+(n-1)d)$$

Arithmetic Mean of two numbers in an AP, p_1 and p_3 is p_2, therefore the arithmetic mean of *m* and *n* can be expressed as: $\tfrac{1}{2}(m+n)$ (pretty obvious really).

Geometric Progressions (GP's)
A GP multiplies each term by an incrementing power of a common ratio, e.g. 12, 6, 3, 1.5 – which the starting number (*a*) is 12, and the common ratio (*r*) is ½, so each term is ar^{n-1}. A GP with *n* terms can therefore be written as:

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

The final term is the general expression for the n^{th} term.
The sum of *n* terms of a GP can be expressed as:

$$S_n = \frac{a(1-r^n)}{1-r}$$

GPs can also be summed to infinity, for $|r| < 1$:

$$S_\infty = \frac{a}{1-r}$$

Geometric Mean of two numbers in a GP, p_1 and p_3, is p_2, as for Arithmetic Mean. The geometric mean of two numbers *m* and *n* can be expressed as: $\sqrt{mn}$

Partial Fractions

These are used for breaking down a fraction with a quadratic denominator. There are three types. Easiest to explain with worked examples.

$$\frac{7x+5}{x^2+x-2} = \frac{7x+5}{(x+2)(x-1)} \equiv \frac{A}{x+2} + \frac{B}{x-1}$$

$\times (x+2)(x-1)$

$\Rightarrow 7x+5 = A(x-1) + B(x+2)$

$x = 1$ to eliminate the A by making the bracket zero

$7+5 = 12 = 3B \therefore B = 4$

$x = -2$ to eliminate B, or alternatively put in B = 4

$-9 = -3A \therefore A = 3$

This is another type of denominator. Once the fraction has been split, the same principle, of eliminating one of the letters by giving *x* a value, to work out the other letter, is used.

$$\frac{4x+5}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \qquad \frac{4x+5}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} \qquad \text{etc.}$$

This rules must also be used one of the factors of the polynomial is in the form $(x - a)^n$, and applying the original rule to the other factor. This is the final type of denominator:

$$\frac{8x^2+4x+1}{(x^2+1)(2x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{2x+1}$$

Calculus

Differentiation

The differential of a function is its rate of change, i.e. its gradient on a graph.

Basic rule of differentiation: $\frac{d}{dx} x^n = nx^{n-1}$

Numerical coefficients remain with the differential.

Product rule and quotient rules:

$\frac{d}{dx} uv = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiating parametric equations and other compound functions – the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Differentiating a function of a function (This is an important principle).

You must multiply the differential of the whole function by the differential of the function inside the function. Confused again? Look at it like this:

$\frac{d}{dx} gf(x) = \frac{d}{dx} g(x) \times \frac{d}{dx} f(x)$

Still confused? Try an example:

$f(x) = ax + b \quad g(x) = x^n$

$y = gf(x) = (ax+b)^n$

$\frac{dy}{dx} = an(ax+b)^{n-1}$

This applies when differentiating trigonometric functions, so $\frac{d}{dx}\sin x^2 = 2x\cos x^2$, and for *e*.
Remember to use the product rule in a situation such as xe^x.
The chain rule can be used in a composite function situation. This is differentiation by substitution, and is easier. The above could be rewritten as:

$y = u^n \qquad u = ax + b$

$\frac{dy}{du} = nu^{n-1} \qquad \frac{du}{dx} = a$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = anu^{n-1} = an(ax+b)^{n-1}$

Do the same when dealing with compound trig functions, e.g. $\sin x^2 = \sin u$, $u = x^2$, differentiate both, and use the chain rule. Alternatively remember the following short-cut rules (but if in doubt, use the chain rule):

$\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$

$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}\left(f(x)\right)^n = nf'(x)\left(f(x)\right)^{n-1}$

The second differential is the differential of the differential, and is written as $\frac{d^2y}{dx^2}$.
A *turning point* exists where the gradient (differential) of a graph is zero. It can be a maximum (gradient goes from positive to zero to negative), a minimum (gradient goes from negative to zero to positive), or a point of inflexion (gradient does not change sign, just reduces and then increases again) – where there is a change in the sense in which the curve is turning (from clockwise to anti-clockwise).
To determine whether a stationary point (where the differential is zero) is a maximum, a minimum or a point of inflexion, look at the differentials (gradients) either side, and work out from there. The second differential can be used to determine whether a stationary point is a maximum or a minimum. The SECOND DERIVATIVE will be NEGATIVE at a MAXIMUM, and POSITIVE at a MINIMUM. If it's zero, use the other method to work out which.

Differentiation of Inverse (Trig) Functions

The following relationship can be applied: $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

Some derivatives of inverse trig functions are given in data book, worth knowing how it works, though, if asked to do a different function or prove it. Example:

$y = \arcsin x \qquad \therefore\ x = \sin y$

$\frac{dx}{dy} = \cos y \qquad \therefore\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}$

Not in terms of x, but $x = \sin y$, so find a way of putting that back in. Use: $\cos^2\theta = 1 - \sin^2\theta$

$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

Differentiation of Implicit Functions
This is the process of differentiating an expression in terms of a term that is not in the equation, e.g. y^2 in terms of x. The following relationship must be used:

$\frac{d}{dx} g(y) = \left(\frac{d}{dy} g(y)\right)\left(\frac{dy}{dx}\right)$

An example is useful:

$$\tfrac{d}{dx} x^2 - y^2 + y = 1$$
$$\Rightarrow \tfrac{d}{dx} x^2 - \tfrac{d}{dx} y^2 + \tfrac{d}{dx} y = \tfrac{d}{dx} 1$$
$$\Rightarrow 2x - 2y \tfrac{dy}{dx} + \tfrac{dy}{dx} = 0$$
$$\therefore 2x = \tfrac{dy}{dx}(2x-1)$$
$$\therefore \tfrac{dy}{dx} = \frac{2x}{(2y-1)}$$

If you have a term that has both *x* and *y* in it, use the product rule on it.
Also note that if you are asked to differentiate a term only, cannot be solved, just leave as: e.g. $\tfrac{d}{dx} y^3 = 3y^2 \tfrac{dy}{dx}$

Integration

Integration is the reverse of differentiation, and finds the area under a graph. In general:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad \int ax^n dx = a\int x^n dx = \frac{ax^{n+1}}{n+1} + c$$

Standard integrals are given in formula book. When doing an integral, if in doubt, differentiate to make sure you get the original back again. In particular, remember from differentiating compound functions, divide by the differential of the secondary function, e.g.:

$\int e^{ax} dx = \frac{e^{ax}}{a}$ Check: $\frac{d}{dx}\left(\frac{e^{ax}}{a}\right) = \frac{ae^{ax}}{a} = e^{ax}$ Also: $\int \cos ax = \frac{\sin ax}{a}$

This 'working backwards' principle is important. Always look out for recognisable derivatives, e.g. because we know that $\tfrac{d}{dx} e^u = \tfrac{du}{dx} e^u$, if we are asked to integrate that derivative, it is simple: $\int \left(\tfrac{du}{dx}\right) e^u dx = e^u + c$ But we must be able to recognise it! Easy to miss so be careful.

Integrating *x* as a power, not given in formula book:

$$\int a^x dx = \frac{a^x}{\ln a}$$

Integration is either definite or indefinite. Definite integration provides a range from which the answer can be calculated, indefinite integration just provides an expression. This illustrates the difference:

$$\int_2^6 x^2 dx = \left[\frac{x^3}{3}\right]_2^6 = \frac{6^3}{3} - \frac{2^3}{3} = \frac{208}{3}$$

$$\int x^2 dx = \frac{x^3}{3} + c$$

When doing indefinite integration, <u>DO NOT forget +c</u> at the end. The value of this constant can be calculated only if some other information is given, which it will be. Common use is for creating an expression for distance travelled from velocity, in which case look out for a statement like "initially at", which is at t=0, which should cancel all the other terms, leaving just the c, which is that value. *Note*: if it doesn't cancel all terms, the *c* is just what's missing.

Trapezium Rule

Integration can be approximated using the trapezium rule, which splits the area under the graph into small trapeziums whose areas are summed. Rule is given in formula book. The equation is given below. *h* is the width of the trapeziums, and each of the *f*'s are the heights of the ordinates at those given points. The more ordinates, the more accurate.

$$\int_{x_m}^{x_n} f(x)dx = \tfrac{1}{2}h\left(f_m + 2(f_{m+1} + f_{m+2} + \ldots + f_{n-1}) + f_n\right)$$

Integration by Substitution

Remember the following principle: $\int f(u)\frac{du}{dx}dx = \int f(u)du$

Although it's not strictly true, the principle here to remember is that multiplying by dx can cancel with the bottom dx, and they can be split so that if $\frac{du}{dx} = x \Rightarrow du = x\,dx$

The idea is to change the whole integral so that an expression for *du* can be made in terms of (*x*)*dx* so that the whole integral can be made wrt u. Confused? Example time…

$$\int (x^2+1)^5 2x\,dx \qquad u = x^2+1$$

$$\int u^5 2x\,dx \qquad \frac{du}{dx} = 2x \qquad du = 2x\,dx$$

$$\int u^5\ du = \frac{u^6}{6} + c = \frac{(x^2+1)^6}{6} + c$$

A common but not immediately obvious situation for this rule is with trig functions, e.g.

$$\int \cos x \sin^n x\,dx = \int \cos x\, u^n\ dx$$

$$u = \sin x \therefore \frac{du}{dx} = \cos x \therefore du = \cos x\,dx$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + c = \frac{\sin^{n+1} x}{n+1} + c$$

This is also useful when integrating fractions, e.g.

$$\int \frac{2x}{\sqrt{x^2+1}}dx = \int \frac{2x}{\sqrt{u}}dx$$

$$u = x^2+1 \qquad \frac{du}{dx} = 2x \qquad du = 2x\,dx$$

$$\int \frac{1}{\sqrt{u}}du$$

The following general rule can be applied to certain situations:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

When doing this with definite integration, REMEMBER THE CHANGE THE LIMITS. To do this, simply put into *u*= the maximum and minimum values of *x*, to give the new maximum and minimum values to be used now the integral is wrt u. e.g. from earlier example:

$$\int_0^3 \frac{2x}{\sqrt{x^2+1}}dx \Rightarrow \int_1^{10} \frac{1}{\sqrt{u}}du = \left[\sqrt{u}\right]_1^{10} = \sqrt{10} - \sqrt{1}$$

$$\because u = x^2+1 \qquad u_{\min} = 0^2+1 = 1$$

$$u_{\max} = 3^2+1 = 10$$

Integrating Fractions

Remember the following principle (it's in the formula book, just don't forget about it!):

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$$

So as long as the top is the differential of the bottom, this can be used immediately. Note that it is not often immediately obvious, but remember that lone numbers are lost during differentiation.
Also REMEMBER: coefficients can be taken out of the integral, e.g.
$\int \frac{4x}{x^2+2}$ cannot have that rule applied to it, but $2\int \frac{2x}{x^2+2}$ can.

Partial fractions can be used to simplify a fraction with a quadratic denominator.

Integration by Parts
This is a special rule, for the integral of a product of two terms, *u* and *v* (e.g. xe^x).
$$\int v \tfrac{du}{dx} dx = uv - \int u \tfrac{dv}{dx} dx + c$$
This looks complicated, so be sure to chose which to be *u* and which to be *v* carefully, so that when used in the integral they become simpler (e.g. make *v*=*x*, so that dv/dx=1). Don't be concerned if you have to do integration by parts again on the answer, e.g. for $x^2\sin x$, but if things don't get any better after that you may have gone wrong, see if they work better if they are assigned the other way around. Look out for other special things, such as the first integral being the same as the second *except* that the sign is different, then the two expressions could be added, eliminating one term.
A special use of integration by parts is to integrate ln*x*:

$\int \ln x dx \qquad v = \ln x \qquad \tfrac{du}{dx} = 1$

$\qquad \tfrac{dv}{dx} = \tfrac{1}{x} \qquad u = x$

$$\int \ln x dx = x \ln x - \int x \tfrac{1}{x} dx = x \ln x - \int 1 dx = x \ln x - x + c = x(\ln x - 1) + c$$

Definite Integration by Parts
Remember to use the appropriate limits for the integrals:
$$\int_a^b v \tfrac{du}{dx} dx = [uv]_a^b - \int_a^b u \tfrac{dv}{dx} dx$$

Improper Integrals (Integrating to Infinity)

Integrating Trig Functions
To integrate sin or cos with an even power, just use the double-angle identities. Examples:
$$\int \sin^2 x\, dx = \int \tfrac{1}{2}(1-\cos 2x)\, dx = \int \tfrac{1}{2} dx - \tfrac{1}{2}\int \cos 2x\, dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$
$$\int \cos^4 x\, dx = \int (\cos^2 x)^2 dx = \int \left(\tfrac{1}{2}(1+\cos 2x)\right)^2 dx = \int \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$
$$= \int \tfrac{1}{4}(1+2\cos 2x) + \tfrac{1}{4}\left(\tfrac{1}{2}(1+\cos 4x)\right) dx = \int \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos 4x$$

If you want to integrate sin or cos with an odd power, it is first split into a power of a power of two, and the power of one. The identity $\cos^2 x + \sin^2 x \equiv 1$ is used for the power of two. e.g.
$$\int \sin^3 x\, dx = \int (\sin^2 x)(\sin x) dx = \int (1-\cos^2 x)(\sin x) dx = \int \sin x - \cos^2 x \sin x\, dx$$

Volumes of Revolution
This is the volume an area would make when it is rotated around the *x*-axis, calculated as:

$\int \pi y^2 dx = \pi \int y^2 dx \qquad$ e.g. $y = e^x \Rightarrow \pi \int (e^x)^2 dx = \pi \int e^{2x} dx$

When rotating around the *y* axis, imagine the *y* axis as the *x* axis, and create the inverse of the function, and integrate that, i.e.

$\pi\int x^2 dy$ e.g. $y = x^2 + 1 \therefore x^2 = y - 1 \Rightarrow \pi\int (y-1)\,dy$

Might help to visualise the shape in odd circumstances, e.g. area between two lines – look at taking away the volumes. Note also easy volumes created by straight lines (triangles/cones).

Differential Equations

An expression of the rate at which one factor (y) increases wrt another (x) (dy/dx =). To find the *general solution* of a first-order differential equation, multiply both sides by *dx* (or whatever), and integrate, e.g.: (note +c may be called +*A* in a diff eqn).

$$\tfrac{dy}{dx} = 4x^2 \qquad dy = 4x^2 dx \qquad \int 1\,dy = 4\int x^2\,dx \qquad y = \frac{4x^3}{3} + c$$

Note that that's a very simple example. Things are often more complicated, but should be fairly easy as long as you *remember to move all x's and y's to the correct sides*, e.g.:

$$\frac{1}{x}\frac{dy}{dx} = \frac{2y}{x^2+1} \quad \Rightarrow \quad (\div y, \times x)\ \frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} \quad \Rightarrow \quad \int y^{-1}\,dy = 2\int \frac{x}{x^2+1}\,dx$$

Remember that the +c can be anything you like, for example it could be a ln, if it's more convenient. For example, here's a little shortcut to remember:

$$\ln y = \ln x + \ln A \quad \Rightarrow \quad \ln y = \ln Ax \quad \Rightarrow \quad y = Ax$$

You may be asked to devise a differential equation. Remember: if the rate is a *decrease* in the factor, you must write it as $-\frac{dy}{dx}$. For example, "the *decrease* is *proportional* to the square root of factor *x*". If proportional, put in constant. It will give info to work out constant. Example:

$\frac{dx}{dt}\,\alpha\sqrt{x}\ \therefore\ -\frac{dx}{dt} = k\sqrt{x}$ Side note: if inversely proportional: $\frac{dx}{dt}\,\alpha\,\frac{1}{\sqrt{x}}$ $\frac{dx}{dt} = \frac{k}{\sqrt{x}}$

$$\Rightarrow -dx = k\sqrt{x}\,dt \quad \Rightarrow -\tfrac{1}{\sqrt{x}}dx = k\,dt \quad \Rightarrow -\int x^{-\frac{1}{2}}\,dx = k\int 1\,dt \quad \Rightarrow -2\sqrt{x} = kt + c$$

Natural (Exponential) Growth (or Decay)

Naturally occurring rate, where rate of growth is proportional to current quantity, i.e.:

$$\tfrac{dQ}{dt} = kQ \qquad \int \tfrac{1}{Q}\,dQ = \int k\,dt \qquad \ln Q + \ln A = kt \qquad \ln AQ = kt$$

$$AQ = e^{kt} \qquad Q = Be^{kt} \qquad (B = \tfrac{1}{A})$$

Exponential decay is reverse, e^{-kt}. Half-life ($t_{1/2}$) = time taken for half of sample to decay, i.e.:

$$\tfrac{1}{2}Q_0 = Q_0 e^{-kt} \quad e^{-kt} = 2 \quad \therefore \quad \frac{\ln 2}{k} = t_{\frac{1}{2}}$$

Vectors

A vector can be written in either of two forms:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

Where ***i***, ***j***, ***k*** are unit vectors in the x, y and z directions respectively.

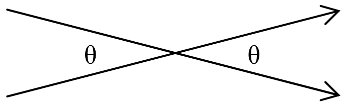
A point vector specifies the co-ordinates of that point from the origin. Vectors can be added:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

The length (modulus) of a vector can be found as follows:
$$\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$|\,\mathbf{a}\,| = a = \sqrt{x^2 + y^2 + z^2}$$

The following equation can be used for finding the angle between vectors:
$$\mathbf{a.b} = x_1x_2 + y_1y_2 + z_1z_2$$
$$\mathbf{a.b} = ab\cos\theta$$
$$\cos\theta = \frac{\mathbf{a.b}}{ab} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^{\,2} + y_1^{\,2} + z_1^{\,2}}\sqrt{x_2^{\,2} + y_2^{\,2} + z_2^{\,2}}}$$

The angle to be found is the one between converging or diverging vectors, as shown in the following diagram:



This equation can be used to identify parallel and perpendicular vectors. Parallel vectors will have an angle of 0° between them, so:
$$\mathbf{a.b} = ab\cos\theta = ab\cos 0° = ab$$
$$\because \cos\theta = \frac{\mathbf{a.b}}{ab} = 1$$
Whereas perpendicular vectors will have an angle of 90°:
$$\mathbf{a.b} = ab\cos\theta = ab\cos 90° = 0$$
$$\because \cos\theta = \frac{\mathbf{a.b}}{ab} = 0$$

Additionally, the coefficients of **i**, **j** and **k** in parallel vectors must be in the same proportions to each other, i.e. **i** + 2**j** + 4**k** is parallel to 2**i** +4**j** + 8**k**.

Skew lines are a pair of line that are not parallel and do not cross each other. Note that there will be an angle between skew lines. They can only exist in three dimensions.

The equation of a line that goes through points A and B can be expressed as follows:
$$A = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad B = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \qquad AB = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$
$$P = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$
The equation is made up of the point it goes through, the direction it goes in, and a scalar (λ).

This is used to find out the equation of the line joining two points, A and B. The point part of the equation can either be A or B, and the direction will be *AB*.

To show that a point (C) is on a given line:

C = (15, -1, 14)

P = 3**i** + 2**j** + 5**k** + λ(4**i** – **j** + 3**k**)

If C is on line P, P = C, i.e.

3**i** + 2**j** + 5**k** + λ(4**i** – **j** + 3**k**) = 15**i** – **j** +14**k**

If that is the case, for each dimension, λ will be the same:

3 + 4λ = 15   λ = 3

2 – λ = –1   λ = 3

5 + 3 λ = 14   λ = 3

As λ is the same for each, it is on the line.

To locate where two lines, P and Q, cross:

P = 3**i** + 2**j** + 5**k** + λ(4**i** – **j** + 3**k**)

Q = 57**i** – 13**j** + 58**k** + μ(17**i** – 5**j** + 19**k**)

Look at them, dimension by dimension:

X's: 3 + 4λ = 57 +17μ

Y's: 2 – λ = –13 –5μ

Z's: 5 + 3λ = 58 +19μ

Convert these to expressions in terms of λ and μ:

54 = 4λ – 17μ

15 = λ – 5μ

53 = 3λ – 19μ

Find two to use as a simultaneous equation:

54 = 4λ – 17μ

60 = 4λ – 20μ

Therefore μ = –2, λ = 5.

Put these values in the vector equations, to get points. If all is correct, the points given should be the same, and this is the point where the lines cross.

Kinematics

There are three properties of a moving object that are dependent upon time and on each other: displacement, velocity and acceleration. There are four equations that can be used to calculate other properties. These are in the formula book:

$v = u + at$

$v^2 = u^2 + 2as$

$s = ut + \tfrac{1}{2}at^2 = vt - \tfrac{1}{2}at^2$

$s = \tfrac{1}{2}(u + v)t$

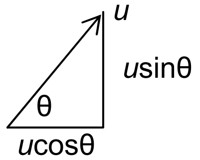
Differentiate (gradient) →

Displacement (s)   Velocity (v)   Acceleration (a)

← Integrate (area)

When moving between those properties with vectors, differentiate/integrate the coefficients of the unit vector directions. Also, remember +c for integrals – and work it out.

Projectiles



Kinematics where a moving object is launched at an angle. No air resistance, so purely kinematics in horizontal direction. Gravity (g) acting as acceleration in vertical direction. *u* in each of those directions given by relevant component in diagram. Look at directions individually.

This table shows the expressions for the kinematics properties in each dimension. u is the relevant components as given in the earlier diagram. There is constant downward acceleration due to gravity, and no air resistance slowing it down horizontally. v is derived from $v = u + at$, and s is derived from $s = ut + \frac{1}{2}at^2$.

	Horizontal (x)	Vertical (y)
u	$u_x = u\cos\theta$	$u_y = u\sin\theta$
a	$a_x = 0$	$a_y = -g$
v	$v_x = u\cos\theta$	$v_y = u\sin\theta - gt$
s	$x = (u\cos\theta)t$	$y = (u\sin\theta)t - \frac{1}{2}gt^2$

Some key facts can be calculated from little information. The following are some useful formulae, with their derivations:

T = time of flight: where $(u\sin\theta)t - \frac{1}{2}gt^2 = 0$, $T = \frac{2u\sin\theta}{g}$

That formula would need modification if the flight ends above $y=0$.

AB = total horizontal distance travelled = $(u\cos\theta)T = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g}$

H = greatest height reached (i.e. where $v = 0$): $0 = u^2\sin^2\theta - 2gy_{max}$ $\quad H = y_{max} = \frac{u^2\sin^2\theta}{2g}$

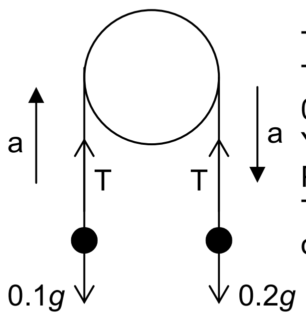
Mechanics

Newton's Laws

Apparently must be "familiar" with these:

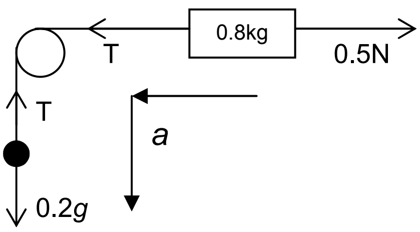
1. A body will remain at rest or in motion with a constant speed in a constant direction unless external forces act upon it.
2. The rate of change of momentum of a body is proportional to the force being applied, in the direction of the force: ***F = ma***
3. The force exerted by one body on another is equal (magnitude) and opposite (direction) to the force being exerted in return.

Simple Pulleys



This is actually very easy. Write two equations:
$T - 0.1g = 0.1a$
$0.2g - T = 0.2a$
You are looking in the *direction* of the *acceleration*.
Put in g and solve simultaneously to get T and a.
Those equations are based on **F = ma** – the force is the resultant of the tension and the downwards accelerations (F = ma again).

Here is another example: a body on a flat surface, with a retarding frictional force of 0.5N. To calculate the overall acceleration, look at objects A (0.2kg) and B (0.8kg) in turn:

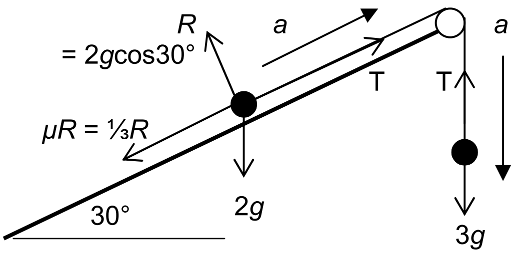


A: F = $0.2g - T = 2 - T = ma = 0.2a$
B: only horizontal is relevant: F = $T - 0.5 = ma = 0.8a$
So you now have the two simultaneous equations, as before:
$0.2a = 2 - T$
$0.8a = T - 0.5$
One way of solving is to add them to get a (= 1.5 ms^{-2}).

With all these things, just use **F = ma** and common sense, and hope it works.

Friction and Sloping Planes

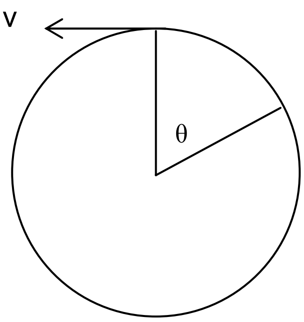
Things get a bit more complicated when friction and slopes are taken into account, but as long as a few rules are remembered it is easy. Frictional force $F = \mu R$ where *μ* is the coefficient of friction between the object and the surface, and *R* is the reaction force of the surface acting on the object, perpendicular to the frictional force. With slopes at an angle, forces get converted from horizontal or vertical to angular through cos/sin and vice versa.



Masses = 2kg, 3kg. *μ* = ⅓.
Resultant perp. to plane: *R* – 2*g*cos30° = 0, *R* = *g*√3
Parallel to plane: T – ⅓*R* – 2*g*sin30° = F = *ma* = 2*a*
Insert *R*: T – ⅓*g*√3 – *g* = T – ⅓*g*(√3 + 3) = 2*a*
For 3kg mass: 3*g* – T = 3*a*
Then do simultaneous equations to find overall *a* and T.

Tip: Remember to note in which direction you are resolving, and make sure all forces acting in that plane are accounted for. Can also have two sloping planes.

Circular Motion



Circular motion is the rotation of a body around a central point. Its direction of motion is always changing. Remember these formulae:
The angle, θ, is known as the angular displacement.
Angular velocity, ω, is the angle moved in one second, i.e.

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} \quad \text{rad s}^{-1}$$

The arc length is rθ, as for other circular properties (see earlier).
The linear velocity is $v = \frac{r\theta}{t} = r\omega \quad \therefore \quad \omega = \frac{v}{r} \quad \therefore \quad \theta = \frac{vt}{r}$

Remember that in one revolution there are 2π radians, to convert between revs and rads.

To maintain the circular motion, the direction of the linear velocity *v* must constantly change. This change comes in the form of a FORCE, and therefore ACCELERATION, TOWARDS THE CENTRE of the circle. This is known as the centripetal acceleration:

$$a = \frac{v^2}{r} = r\omega^2$$

Some problems will just be a satellite or similar orbiting a body, but others may include a string, attaching the body (of mass *m*) to the centre of the circle. In which case, the tensional force in the string can be calculated as follows (where *r* is the string length):

$$F = ma = mr\omega^2$$

The key equations are simply: $\omega = \frac{\theta}{t} \qquad v = r\omega \qquad a = r\omega^2$

Impulse and Momentum

Momentum = *mv* (momentum is a vector quantity, as is *v*, units are Ns)
In a collision, the total momentum of the bodies is the same before as after.
Impulse = change in momentum that a force produces (*mv – mu)*
Impulse = *Ft* (for constant force, acting for time *t*)
= $\int_0^T F\,dt$ (for variable force, acting for time *T*)

Force = rate of change of momentum of a body (with mass *m*): $F = m\frac{dv}{dt}$ (*F* = *ma*)

Elastic Springs and Strings – *Hooke's Law*
I cannot find this in the syllabus, but we were taught it. Briefly, the tensional force required to extend a string or extend/compress a string by *x* is given by:

$$T = \lambda \frac{x}{a}$$

a = natural (unextended) length
x = extension (or compression for springs)
λ = modulus of elasticity:

$\lambda = \frac{Ta}{x}$ λ represents the tension required to double the length of the string ($x = a$)

Probability and Statistics
Notation: Two events, A and B:

P(not A) = 1 – P(A)
P(A and B) = P(A ∩ B) = P(A) × P(B|A) = P(B) × P(A|B)
P(A or B) = P(A U B) = P(A) + P(B) – P(A and B)
P(B|A) = P(B given that A has occurred) = P(A and B) ÷ P(A)

If two events are *independent*, P(B|A) = P(B), as the occurrence of A has no effect on the occurrence of B, so P(A and B) = P(A) × P(B).

If two events are *mutually exclusive*, they cannot happen together, so P(A and B) = 0, and P(B|A) = P(A|B) = 0, e.g. a dice (can't get a 5 and a 6!), so P(A or B) = P(A) + P(B).

Most is obvious, just needs thinking about. Good example is that P(at least one six) when throwing a dice can be easily expressed as 1–P(no sixes). P(at least two sixes) = 1 – P(one six) – P(no sixes). Venn diagrams can be handy for visualising multiple events probability.

Permutations and Combinations
A permutation is an ordered arrangement of items.
A combination is an unordered group of items.
i.e. Three different items (A, B, C) has six arrangements, but only one combination, i.e.

$${}^{3}P_{3} = 6 \qquad {}^{3}C_{3} = 1$$

$${}^{n}P_{r} = \frac{n!}{(n-r)!} \qquad {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

n gives the total number of items, from which r are selected.

Some puzzles get complicated and need to be thought out. Basic principle with permutations of a set number is simply *x!* – this is because there are *x* ways of choosing the first number, *x-1* for the next, *x-2* for the next, and so on, basically 3×2×1 ways of arranging 3 things.

Statistics
A discrete random variable exists where there are distinct events, each of which has a given probability. The sum of the probabilities of these events = 1 if it is a discrete RV. A probability distribution looks like this:

x	0	1	2	3	4
$P(X = x)$	1/12	2/12	1/12	5/12	3/12

$\sum P(X = x) = 1$ so this is a discrete random variable.

The *expected value* E(X) is the same as the mean. It is calculated by multiplying each P in the above table by *x*, and summing them, i.e.:

$$\sum\left(x \times P(X = x)\right)$$

The expectation of any value can be calculated in the same method, e.g.:

$$E\left(g(x)\right) = \sum\left(g(x) \times P(X = x)\right)$$

$$E(X^2) = \sum\left(x^2 \times P(X = x)\right)$$

Rules of expectations are as follows:

$$E(a) = a$$

$$E(aX) = aE(X)$$

$$E(aX + b) = aE(X) + b$$

The variance, Var(X) [V(X) in the formula book], is calculated as follows:
(σ is the standard deviation)

$$Var(X) = \sigma^2$$

$$Var(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$$

The *Binomial Distribution* is a probability distribution for an event with two outcomes. The values can be looked up in the tables in the formula book. It has two parameters, n and p. n is the total number of events (e.g. coin tossed 10 times, 10 = n), while p is the probability of the event (e.g. a head) occurring. r is the number of events that you want to happen. The value looked up is $P(R \leq r)$, i.e. the probability that the event has occurred r or less times. For the binomial distribution, the following rules for E(X) and Var(X) are true:

$$E(X) = np$$

$$Var(X) = npq = np(1 - p)$$

A *Continuous Random Variable* does not have discrete values, rather the value can exist anywhere within a range (e.g. height, weight). The probability for a given value is given by the *Probability Distribution Function*, f(x).

$$\int_a^b f(x)\,dx = 1$$

Expectations and the Variance of the PDF is calculated in the same way as for a discrete random variable, i.e.

$$E(X) = \int_a^b xf(x)\,dx = \mu$$

$$E\left(g(x)\right) = \int_a^b g(x)f(x)\,dx$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \int_a^b x^2 f(x)\,dx - \mu^2$$

The integral of f(x) is the *Cumulative Distribution Function*, F(x). It indicates the probability of the given value, or less, i.e.

$$F(x) = P(X \leq x) = \int f(x)\,dx$$

F(x) = 0.5 at the median.
The mode is the highest point.

To find out the probability between two points on a CDF:

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

The *Normal Distribution* is the special 'natural' PDF. Use the tables to look up the values for the P(X ≤ x) (i.e. the CDF). Its parameters are the mean μ and the variance σ^2. The standardising formula must be used to convert any mean and standard deviation to the normal distribution. Therefore, this is the figure to look up in the table:

$$P\left(Z \le \frac{X-\mu}{\sigma}\right)$$

Where X is the real value, σ the standard deviation, and μ the mean.

DRAW A DIAGRAM!
A diagram will make anything that is not P(Z ≤ z) make sense, for example:
P(Z ≥ z) = 1 – P(Z ≤ z)
P(a ≤ Z ≤ b) = P(Z ≤ b) – P(Z ≤ a)

The normal distribution can be used to calculate the binomial distribution.
We know that np = μ, and npq = σ^2. These are the parameters we need for the normal distribution. The values can be put into the above standardising formula to find out the figure to look up to represent the binomial distribution.

Averages
Mean = (sum of all values ÷ number of values)
Median = middle value = $((n+1)\div 2)^{th}$ item of a series with n items in total. When a half (when n is even), is mean of items either side, i.e. $12\frac{1}{2}^{th} = ((13^{th}+12^{th})\div 2)$
Mode = item with greatest frequency.

From a frequency table, the mean is the sum of (the values × the frequencies) ÷ total, e.g.

Num heads, *x*	0	1	2	3
Frequency, *f*	6	18	38	23
Cumulative Frequency	6	24	62	85

Mean = $\dfrac{(0\times 6)+(1\times 18)+(2\times 38)+(3\times 23)}{85} = \bar{x}$

When dealing with continuous data (ranges), take mid point of each group, e.g.
2 – 2½ = 40
2½ – 3 = 200

Mean = $\dfrac{(2.25\times 40)+(2.75\times 200)}{240}$

So in summary: $\bar{x} = \dfrac{\sum fx}{\sum f}$

Mode is highest frequency (duh). This is all very much like stuff for RV's.
Median on a cum. freq. graph is the point at half the cum. freq. This can also be estimated from a table by finding half the cumulative frequency.
Upper and lower quartiles are the values at ¼ and ¾ the cum. freq, with middle being IQR.
Percentiles are values at *x*% the cumulative frequency.

Standard Deviation
Calculated as follows:

$$\sqrt{\frac{\sum x_i^{\ 2}}{n} - \bar{x}^2}$$

Where Σx_i^2 is the sum of all the squares items in the range, divided by the number of those items. Subtract the square of the mean, and root it all.

This can be adapted for discrete or continuous frequency tables:

$$\sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

As an example, for the example frequency table given above:

$$\sqrt{\frac{(0^2 \times 6) + (1^2 \times 18) + (2^2 \times 38) + (3^2 \times 23)}{85} - \bar{x}^2}$$

Representation of Statistics

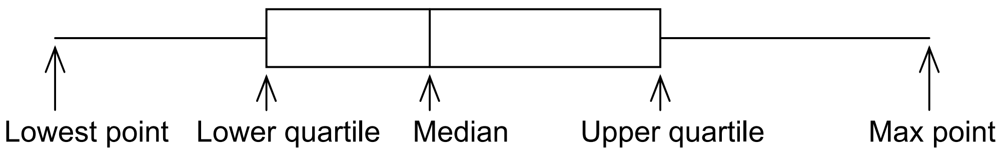
Histograms

- Each group is represented by a bar, the *area* of which represents the frequency of items in the group.
- The width of the bar is the width of the class
- Height of bar is (frequency ÷ class width) = the frequency density

Frequency Polygons are lines joining midpoints of top of histogram bars.
Cumulative Frequency Polygons add up the frequencies of all previous items.

Stem and Leaf Diagrams
'Stem' is the tens, 'leafs' are lines of the units.

Box-and-Whisker Plots



Remember a scale is needed on the bottom.

These can show skewness, which is a measure of the symmetry of the distribution. For a positive skew, the right tail (high values) would be much longer than the left, and/or the upper quartile is further from the median than the lower quartile. Opposite for negative skew.