ELEC 3035, Lecture 7: Observer design
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• Observers

• Observer design by pole placement

• Duality between observer and controller design

• Pole placement by output feedback
General observer design problem

Given dynamical system $B_{\text{ext}}$ with two types of external variables:

- observed variables $w$
- to-be-estimated variables $z$

find system (called observer) accepting $w$ and producing $z$

We will consider the case: $B = B_{i/s/o}(A, B, C, D)$, $w = (u, y)$, $z = x$.

Lecture 6: nonrecursive, feedforward observer for the initial state $x(0)$

Now our goal is recursive feedback observer for the current state $x(t)$
Output feedback control
separation and certainty principles

We can extend a given state-feedback controller

\[ u = Kx \]

to output feedback controller

\[ u = K\hat{x} \]

by using the observer state estimate \( \hat{x} \) in place of \( x \).
Internal model and feedback principles

The observer design is based on the following principles:

1. **Internal model:** the model run by $u$, gives an estimate $\hat{x}$ for $x$

2. **Feedback:** correct the estimate $\hat{x}$, so that the error

$$x(t) - \hat{x}(t) =: e(t) \to 0 \quad \text{as} \quad t \to \infty$$

Let the feedback be a linear function of the output error

feedback correction $= L(y - \hat{y})$

Then the observer for the model $\mathcal{B}_{i/s/o}(A, B, C, D)$ is

$$\sigma \hat{x} = A\hat{x} + Bu - L(y - \hat{y})$$

$$\hat{y} = C\hat{x} + Du$$
Error dynamics

Our goal is to choose $L$, so that the state error $e(t) \to 0$ as $t \to \infty$.

The dynamics of $e$ is

$$\sigma e = \sigma(x - \hat{x})$$
$$= Ax + Bu - A\hat{x} - Bu + L(y - \hat{y})$$
$$= A(x - \hat{x}) + LC(x - \hat{x})$$
$$= (A + LC)e$$

i.e., $e \in \mathcal{B}_{ss}(A_o)$ — an autonomous LTI system.

Therefore, $e(t) \to 0$ as $t \to \infty$ is equivalent to stability of $\mathcal{B}_{ss}(A_o)$. 
Comparison with state-feedback stabilization

In the state feedback stabilization problem we have

\[ \sigma x = Ax + Bu \quad \text{and} \quad u = Kx \]

which gives an autonomous LTI closed loop system

\[ \sigma x = \begin{pmatrix} A + BK \\ \hat{A}_c \end{pmatrix} x \]

and the aim is to choose \( K \), so that \( B_{ss}(A_c) \) is stable.
Observer design by pole placement

The condition $e(t) \to 0$ as $t \to \infty$ is a minimum requirement.

In fact we want $e(t) \to 0$ fast

(possibly in a finite (small) number of steps $\iff$ deadbeat observer)

The error dynamics is governed by the poles of the matrix

$$A_0 := A + LC$$

so for desired error dynamics we can

**select desired pole locations of $A_0$ and choose $L$ to achieve them.**
Duality of the observer PP and controller PP problems

Observer PP problem: Choose $L$, so that

$$\det(zI - (A + LC)) = p_{\text{des}}(z)$$

Controller PP problem: Choose $K$, so that

$$\det(zI - (A + BK)) = p_{\text{des}}(z)$$

Observer PP is not a new problem:

$$\det(zI - (A + LC)) = \det\left(\left(zI - (A + LC)\right)^\top\right)$$
$$= \det\left(zI - (A^\top + C^\top L^\top)\right)$$
$$= \det\left(zI - (\tilde{A} + \tilde{B}\tilde{K})\right)$$

$\Rightarrow$ observer PP is controller PP for the dual system.
The results for state feedback PP can be restated for observer PP:

**Theorem:** The eigenvalues of $A + LC$ can be assigned choosing $L$ to any locations in $\mathbb{C}$ if and only if $A, C$ is observable.

Observer canonical form $\leftrightarrow$ Controller canonical form

**Lemma:**
- Let $A, c$ and $A', c'$ be two observable pairs and
- assume that $A$ and $A'$ have the same char. polynomials.

Then there is a unique similarity transformation given by the matrix

$$T := ( \mathcal{O}(A', c'))^{-1} \mathcal{O}(A, c)$$

such that

$$T^{-1} AT = A' \quad \text{and} \quad cT = c'.$$
Closed-loop system with output feedback controller

Consider the closed loop system

\[
\begin{align*}
\text{Plant:} & & \sigma x = Ax + Bu, & y = Cx + Du \\
\text{Observer:} & & \sigma \hat{x} = A\hat{x} + Bu - L(y - C\hat{x} - Du) \\
\text{State feedback controller:} & & u = K\hat{x}
\end{align*}
\]
Feedback controller:

\[
\sigma \hat{x} = (A + LC)\hat{x} + (B + LD)u - Ly, \quad u = K\hat{x} \\
= (A + LC + BK + LDK)\hat{x} - Ly
\]

Note: the feedback controller is a dynamical system

Closed-loop system:

\[
\sigma \begin{bmatrix} \hat{x} \\ \hat{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + LC + BK \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\hat{x}} \end{bmatrix}
\]

Note: closed-loop system order = plant order + controller order

Error equation:

\[
\sigma \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}
\]
Example: output feedback deadbeat control

15th order single-input open-loop system, 30 order closed-loop system

(The same system as the one used in the example of Lecture 1)