Multi-unit Auctions With Asymmetric Bidders

Ioannis A. Vetsikas and **Nicholas R. Jennings**[1]

Abstract. In the existing work on auctions, a symmetric model of all bidders has been assumed: they all have the same risk attitude, possible valuations and spite (or competition). As this is not realistic in many real examples, we relax this assumption and extend the state of the art in multi-unit auctions to consider bidders of different types; each bidder type has a different risk attitude and distribution from which its valuation is drawn or different spite. We examine both the case when the participants' types are known and when they are not, and, for both cases, derive equilibrium bidding strategies in both m^{th} and $(m+1)^{th}$ price sealed-bid auctions.

1 Introduction

Auctions have become commonplace and they are used to trade all kinds of commodity both between governments, companies, and private individuals. Game theory is widely used in such multi-agent scenarios, as a way to model and predict the behavior of bidders participating in these auctions. The scenarios normally analyzed in the auction literature (and other related disciplines) assume in almost all cases that the bidders participating in the auction are symmetric in the sense that they have their parameters (i.e. their valuations for the goods they bid to buy) drawn from the same prior distributions and have the same utility model. There is very little work that relaxes this assumption. More specifically, [6] and [4] compute equilibria for auctions with asymmetric bidders with different prior distributions from which their valuations are drawn, and an experimental evaluation is conducted in [2].

In this paper, we examine two important multi-unit auction scenarios that have been looked at in the auction literature: the case of bidders with any risk attitude, and of competitive bidders. In the existing work, a symmetric model of all bidders has been assumed; (i) they all have the same distribution from which their valuation is drawn and the same risk attitude thus using the same utility function, and (ii) they all have the same competitiveness respectively. For example, in [5] and [3], cases where agents are not risk neutral, but rather risk averse, are examined. In all instances, the agents are assumed to be risk averse in exactly the same way, and they all have the same utility function, which maps profit to utility in exactly the same way. In [1] and [9] a different kind of utility function is assumed; the bidders in this case not only wish to maximize their own profit, but they also wish to minimize the opponents' profit; these two goals are weighted by the agent's spite coefficient, which determines the relative importance assigned to these goals. In all this literature, the model of all agents is the same, in the sense that they all use the same valuation distribution function, the same utility function, and the same spite coefficient.

To extend these results, we introduce asymmetries in the bidders' models. However, unlike in [4], where the models of all bidders are common knowledge, in this paper we examine not only this case, but also the one where we assume that each bidder only knows his own model: a bidder knows how competitive he is, but not how competitive the opponents are; he does know however that there is a certain chance associated with each opponent using a particular model (i.e. competition coefficient in this case).

This paper is organized as follows. In the next section, we formally present the model and the notation that will be used in this paper. Then, in section 3, we derive the systems of differential equations that characterize the Bayes-Nash equilibria that exist in the case of bidders with different risk attitudes and valuation distributions; first we analyze the case when the opponent models are not known and then the case when they are known to all participants. We also present a specific example to illustrate how to compute the equilibrium numerically; for the remaining cases we point the reader to the algorithm we presented in [10]. In section 4, we do the same for the case when the bidder spite (or competitiveness) is not the same for all bidders. Finally, we conclude.[2]

2 The Multi-Unit Auction Setting

In this section we formally describe the auction setting to be analyzed and define the objective function that the agents wish to maximize. We also give the notation that we use.

In particular, we will compute Bayes-Nash equilibria for sealed-bid auctions where $m \geq 1$ identical items are being sold; these equilibria will be defined by a set of strategies $g_{\alpha_i}(v)$, which map the agents' valuations v_i to bids b_i. These strategies are parameterized by a parameter α_i, which will indicate the model of agent i, i.e. his risk attitude and type of valuations or his spite. Thus we assume that two agents will use the same bidding strategy, if they have the same model (same parameter α_i). The final price is determined by the m^{th} price rule, according to which the top m bidders win one item each at a price equal to the m^{th} highest (last winning) bid respectively.

More specifically, we assume that N indistinguishable bidders (where $N \geq m$) participate in the auction and each has a private valuation (utility) v_i for acquiring any one[3] of the traded items, which is known only to himself; these valuations are assumed to be independent drawn from a distribution with cumulative distribution function (cdf) $F_{\alpha_i}(v)$, which depends on the bidder's model α_i. Furthermore, we assume that $F_{\alpha_i}(v)$ has support in $[v_i^L, v_i^H]$, which means that

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[1] University of Southampton, UK, email: {iv,nrj}@ecs.soton.ac.uk

[2] We would like to point out that in the workshop paper [10] we presented some initial work analyzing the cases when the opponent models are not common knowledge. In this paper, we include some corrections and simplifications of that initial work, as well as the case when the models of all participants are known; furthermore, we extend the first case examined to include not only asymmetric risk attitudes, but also asymmetric valuations.

[3] We make the assumption here that each bidder is interested in exactly one item; this is a usual assumption made in the analysis of multi-unit auctions, as the analysis even for self-interested risk-neutral bidders which are interested in purchasing multiple copies of the item is an open problem.

$\forall v < v_i^L \vee v > v_i^H : F^{'}_{\alpha_i}(v) = 0$.[4] Let $\widetilde{u}_i$ be the profit of agent i (i.e. $\widetilde{u}_i = 0$, if it does not win an item, and $\widetilde{u}_i = v_i - p_i$, if it does) and p_i is the total payment the agent must make to the auctioneer. The agents have varying risk attitudes. The possible risk attitudes belong to a family of utility functions $u_{\alpha_i}()$, which are characterized by the type (model) α_i of each agent. Thus, we assume that the objective function (i.e. the total utility) that each agent tries to maximize depends only on his own gain $\widetilde{u}_i$, and is equal to:

$$U_i = u_{\alpha_i}(\widetilde{u}_i)$$

Some families of utility functions $u_\alpha(x)$ used widely in economics are: $u_\alpha(x) = x^\alpha, \alpha \in (0,1)$ (CRRA), and $u_\alpha(x) = 1 - exp(-\alpha x), \alpha > 0$ (CARA); both characterize risk-averse bidders.

This model is the first scenario we analyze (in section 3), where we examine self-interested agents with different risk attitudes and valuation distributions. We examine two cases for this scenario. In the first, each agent has uncertainty not only for the opponents' valuations v_j, but also for their model (i.e. their parameters α_j; they know the prior distribution $h(\alpha)$ from which each opponent parameter α is drawn, which is the probability that each participant is of a particular type α.[5] So even though each agent knows only it's own model, it can make a probabilistic inference on the possible opponent models. In the second case examined, the models of all participants are known.

In the second scenario, the agents are now risk-neutral and their valuations are drawn from the same distribution, but all agents are competitive, rather than self-interested. This means that they not only care about maximizing their profit, but also about minimizing the profit of the opponents. We define the objective function of each agent in the same way as in [1, 9]:

$$U_i = (1-\alpha_i)\widetilde{u}_i - \alpha_i \sum_{j \neq i} \widetilde{u}_j$$

where $\alpha_i \in [0,1]$ is a parameter called the *competition (or spite) coefficient*, which denotes the degree of competition of agent i; the higher it is the more the agent cares about minimizing the opponent profit, rather than maximizing his own.

We also use the following additional notation in the proofs:

$$\Phi_k(x) = \sum_{i=0}^{k-1} C(N-1,i) x^{N-1-i}(1-x)^i \tag{1}$$

where the notation $C(n,k)$ is the total number of possible combinations of k items chosen from n. This formula is useful because, if $Z(x)$ is the probability distribution of any opponent's bid b_j, i.e. $Z(x) = Prob[b_j \leq x]$, and $B^{(k)}$ is the k^{th} order statistic of these bids of the opponents, then the distribution of $B^{(k)}$ is: [8]

$$Prob[B^{(k)} \leq x] = \Phi_k(Z(x)) \tag{2}$$

$\forall N, m$, such that $N \geq m$ the following equations hold: [9]

$$\Phi'_m(x) = (N-m)\big(\Phi_m(x) - \Phi_{m-1}(x)\big)\frac{1}{x} \tag{3}$$

$$\Phi'_m(x) = m\big(\Phi_{m+1}(x) - \Phi_m(x)\big)\frac{1}{1-x} \tag{4}$$

We will use equations 1 through 4 in the computation of the equilibria. To reduce the size of some equations in the proofs, let us also define:

$$\Delta\Phi_m(x) = \Phi_m(x) - \Phi_{m-1}(x) \tag{5}$$

We also need the following definition which will be used in the proofs of the equilibria when the models of all participants are common knowledge:

Definition 1 *Given a set S (with all its elements being unique), let us define the k-subset $S^{(k)}$ of S to be the subset of its powerset 2^S whose elements have cardinality $k \leq |S|$. More formally:*

$$S^{(k)} = \{s \in 2^S : |s| = k\}$$

For the specific case of the set $S = \{1, \ldots, m\}$, let us define:

$$P_{k,m} = \{1, \ldots, m\}^{(k)}$$

which is the set containing all the possible ways of selecting k different numbers out of the set of numbers 1 through m. Finally, we define the extensions of this definition:

$$P_{k,m}^{-\{i\}} = (\{1, \ldots, m\} - \{i\})^{(k)}$$

$$P_{k,m}^{-\{i,j\}} = (\{1, \ldots, m\} - \{i,j\})^{(k)}$$

which are the k-subset of all the numbers 1 through m without counting any subsets containing i (and i, j respectively).

3 Asymmetric Valuations and Risk Attitudes

In this section, we assume that agents have asymmetric valuations and risk attitudes. In section 3.1 we present the equations that characterize the equilibria when each bidder doesn't know the models of his opponents, while in section 3.2, we present the same analysis when the models of all participants is common knowledge. At the end of section 3.1, we give a specific example of how to derive the equilibrium strategies (for the case of theorem 1) numerically.

3.1 The Opponent Models Are Not Known

In this section, we assume that each agent has uncertainty not only for the opponents' valuations, but also for their models (i.e. risk attitudes and distributions of valuations). The possible risk attitudes and distributions of valuations belong to a family of functions, which are characterized by an one dimensional parameter α, which is drawn from a known probability distribution (h). We therefore assume that each agent i knows its own valuation v_i, risk attitude function $u_{\alpha_i}()$ and the distributions $F_{\alpha_i}(v)$, as well as the distribution $h(\alpha)$ from which models of the opponents, meaning the risk attitude functions $u_\alpha()$ and distributions $F_\alpha(v)$, are drawn. We assume that the number of possible models are λ, meaning that the possible models are characterized by $\alpha = \alpha_1, \ldots, \alpha_\lambda$.

We initially present the system of equations that characterize the equilibrium and then show how to solve them.

Theorem 1 *In the case of an m^{th} price sealed-bid auction with N participating bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, which is drawn from $F_{\alpha_i}(v)$, and has a risk attitude described by utility function $u_{\alpha_i}()$, both of which describe his model α_i (where α_i are i.i.d. random variables drawn from distribution $h(\alpha)$), strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations:*

$$\forall x, \alpha_i : (N-m) \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F'(\zeta_\alpha(x))\zeta'_\alpha(x)h(\alpha) = \tag{6}$$

$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x) - x)}{u_{\alpha_i}(\zeta_{\alpha_i}(x) - x) - u_{\alpha_i}(0)} \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(\zeta_\alpha(x))h(\alpha)$$

with boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L$ for all i such that $v_i^L = \min_j\{v_j^L\}$. There are λ possible bidder models characterized by parameter $\alpha = \alpha_1, \ldots, \alpha_\lambda$.

Proof. We assume that the equilibrium strategy is described by functions $g_\alpha(v)$ which map the valuations v to bids for any of the possible risk attitude functions $u_\alpha()$. We use this knowledge to determine the bids of the opponents and the expected profit that a bidder i gets from placing a bid equal to b_i. The distribution from which an opponent's bid b_j is drawn has cdf: $Prob[b_j \leq x|\alpha_j] = F_{\alpha_j}(g_{\alpha_j}^{-1}(x))$, when his risk attitude is described by function $u_{\alpha_j}()$. Therefore, using Bayes' rule we compute this probability for any possible value of α_j:

$$Prob[b_j \leq x] = \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(g_\alpha^{-1}(x))h(\alpha) \tag{7}$$

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[4] For example, if $F_{\alpha_i}(v)$ is the uniform distribution $U[0,1]$, then $v_i^L = 0$ and $v_i^H = 1$. If $F_{\alpha_i}(v)$ is such that v_i can take values in $[0, +\infty)$, then $v_i^L = 0$ and $v_i^H = +\infty$. We use these tight bounds to define the boundary conditions of some of the equilibria we will compute.

[5] Thus $h(\alpha)$ is the probability that the opponent is of type α and this distribution is assumed unique, although the results can be extended to continuous distributions as well, as shown in [10].

The distribution of the k^{th} highest opponent bid $B^{(k)}$, as there are $(N-1)$ opponents, is:

$$Prob[B^{(k)} \leq x] = \Phi_k\Big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(g_\alpha^{-1}(x))h(\alpha)\Big) \tag{8}$$

where $\Phi_k(x)$ is given by equation 1.

We can now analyze the expected profit of bidder i. Let b_i be the bid that he places in the auction. We distinguish the following cases:
(i) If $b_i < B^{(m)}$, then bidder i is outbid and doesn't win any items, therefore his utility is $u_i = u_{\alpha_i}(0)$.
(ii) If $B^{(m)} \leq b_i \leq B^{(m-1)}$, then bidder i has placed the last winning bid. Thus the payment equals his bid and his utility is $u_i = u_{\alpha_i}(v_i - b_i)$. The probability of this case happening is: $Prob[B^{(m)} \leq b_i \leq B^{(m-1)}] = \Delta\Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big)$.
(iii) If $B^{(m-1)} < b_i$, then bidder i is a winner, the payment is equal to bid $B^{(m-1)}$ and his utility is $u_i = u_{\alpha_i}(v_i - B^{(m-1)})$. Note that: $Prob[B^{(m-1)} \leq \omega] = \Phi_{m-1}\Big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(g_\alpha^{-1}(\omega))h(\alpha)\Big)$.

The expected utility of bidder i, who places bid b_i, is:

$$\begin{aligned} EU_i(b_i) =& u_{\alpha_i}(0)\Big(1 - \Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big)\Big) \\ &+ u_{\alpha_i}(v_i - b_i)\Delta\Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big) \\ &+ \int_0^{b_i} u_{\alpha_i}(v_i - \omega)\frac{d}{d\omega}\Big(\Phi_{m-1}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(\omega))h(\alpha)\big)\Big)d\omega \end{aligned} \tag{9}$$

The bid which maximizes this expected utility, is found by setting: $\frac{dEU_i}{db_i} = 0$. This becomes:

$$\begin{aligned} &(u_{\alpha_i}(v_i - b_i) - u_{\alpha_i}(0))\frac{d}{db_i}\Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big) \\ &= u'_{\alpha_i}(v_i - b_i)\Delta\Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big) \end{aligned} \tag{10}$$

Thus using equation 3 to simplify equation 10, we derive:

$$(N-m)\frac{\frac{d}{db_i}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big)}{\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)} = \frac{u'_{\alpha_i}(v_i - b_i)}{u_{\alpha_i}(v_i - b_i) - u_{\alpha_i}(0)}$$

This value b_i is equal to $b_i = g_{\alpha_i}(v_i)$, since it maximizes the expected utility $EU_i(b_i)$. Using this substitution, we derive the system of differential equations:

$$\begin{aligned} \forall v_i, \alpha_i: \ (N-m) \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} \frac{F'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}h(\alpha) = \\ \frac{u'_{\alpha_i}(v_i - g_{\alpha_i}(v_i))}{u_{\alpha_i}(v_i - g_{\alpha_i}(v_i)) - u_{\alpha_i}(0)} \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))h(\alpha) \end{aligned} \tag{11}$$

for all possible values of v_i, α_i. The boundary conditions come from the fact that a bidder with the lowest possible valuation that any bidder can have $v_i = v_i^L$ will always bid $b_i = v_i^L$.

Now, to simplify these equations we make the following substitutions:
(i) As the equations hold for all $\forall v_i, \alpha_i$, therefore, if we set a new variable $x = g_{\alpha_i}(v_i)$, which takes values in $x \in [g_{\alpha_i}(v_i^L), g_{\alpha_i}(v_i^H)]$, we transform the equations to the following:

$$\begin{aligned} \forall x, \alpha_i: (N-m) \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} \frac{F'(g_\alpha^{-1}(x))}{g'_\alpha(g_\alpha^{-1}(x))}h(\alpha) = \\ \frac{u'_{\alpha_i}(g_{\alpha_i}^{-1}(x) - x)}{u_{\alpha_i}(g_{\alpha_i}^{-1}(x) - x) - u_{\alpha_i}(0)} \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(x))h(\alpha) \end{aligned} \tag{12}$$

(ii) By setting $\zeta_{\alpha_i}()$ to be the inverse function of $g_{\alpha_i}()$, the equation becomes the system of equations 6. ■

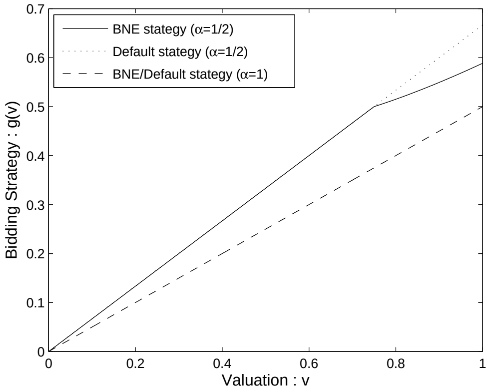

Figure 1. Equilibrium strategies $g(v)$ for an m^{th} price auction in which bidders use the CRRA utility function $u(x) = x^\alpha$ and there is a 50% chance that the opponents are risk neutral ($\alpha = 1$) and a 50% chance that they are risk averse with $\alpha = \frac{1}{2}$. Also included are the default equilibrium strategies (all the opponents have the same competition factor α as the bidder); in the case when $\alpha = 1$, the default strategy is the same of the BNE strategy, so it is not graphed separately. The valuations v are drawn from the uniform distribution $U[0,1]$. The number of bidders is $N = 3$ and the number of items being sold is $m = 2$.

Computing the Equilibrium Strategies The equations 6 seem quite complex. Thus, we show in this section how to solve them. We assume that α_i are ordered based on the value of v_i^L, meaning that we order them so that $v_1^L \leq \ldots \leq v_\lambda^L$. This assumption is crucial for the following steps to work:
(i) In order for this system to have a solution it must be:

$$\frac{u'_{\alpha_1}(\zeta_{\alpha_1}(x) - x)}{u_{\alpha_1}(\zeta_{\alpha_1}(x) - x) - u_{\alpha_i}(0)} = \ldots = \frac{u'_{\alpha_\lambda}(\zeta_{\alpha_\lambda}(x) - x)}{u_{\alpha_\lambda}(\zeta_{\alpha_\lambda}(x) - x) - u_{\alpha_\lambda}(0)} \tag{13}$$

This gives $(\lambda - 1)$ independent equations; differentiating each one of these gives us the following:

$$\zeta'_{\alpha_i}(x) = 1 + (\zeta'_{\alpha_1}(x) - 1) \cdot \tag{14}$$

$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x) - x)u'_{\alpha_1}(\zeta_{\alpha_1}(x) - x) - u''_{\alpha_1}(\zeta_{\alpha_1}(x) - x)(u_{\alpha_i}(\zeta_{\alpha_i}(x) - x) - u_{\alpha_i}(0))}{u'_{\alpha_1}(\zeta_{\alpha_1}(x) - x)u'_{\alpha_i}(\zeta_{\alpha_i}(x) - x) - u''_{\alpha_i}(\zeta_{\alpha_i}(x) - x)(u_{\alpha_1}(\zeta_{\alpha_1}(z) - z) - u_{\alpha_1}(0))}$$

which is used to substitute all $\zeta'_{\alpha_i}()$ with terms containing only $\zeta'_{\alpha_1}()$ in equation 13. Thus we derive a differential equation $\zeta'_{\alpha_1}()$ is equal to a function of $\zeta_{\alpha_i}(), \forall i$, where $\zeta_{\alpha_i}()$ can be computed from $\zeta_{\alpha_1}()$ using equation 13. This is solved by using the a standard Runge-Kutta method, whose algorithm is presented in chapter 17 of [7], with one modification: the values of $\zeta_{\alpha_i}(), i = 2, \ldots, \lambda$ are computed at each step from the values of $\zeta_{\alpha_1}()$ solving equation 13 using the Bisection Method; see chapter 9 of [7] for this algorithm.
(ii) Because in step 1, x is defined for $x \in [g_{\alpha_i}(v_i^L), g_{\alpha_i}(v_i^H)]$, we need be careful when $\zeta_{\alpha_i}(x) > v_i^H$ or $\zeta_{\alpha_i}(x) < v_i^H$ for any i. For such values, it is $F(\zeta_{\alpha_i}(x)) = 0$ and $F(\zeta_{\alpha_i}(x)) = 1$ respectively and also $F'(\zeta_{\alpha_i}(x)) = 0$. When performing the simplification of the previous step, we need to keep in mind this fact and that the equations 13 only hold for values of x such that $\zeta_{\alpha_i}(x) \in [v_i^L, v_i^H]$.

Example We give now a simple example of asymmetric risk attitudes. We examine an m^{th} price auction, with $N = 3$ bidders and $m = 2$ items for sale, where there are two possible models of bidders using the CRRA utility function $u_\alpha(x) = x^\alpha$, one where $\alpha = 1$ (risk neutral bidder) and another where $\alpha = 0.5$ (risk averse), both with probability 50%. In this example we have the following system of equations (obtained from equations 13 and 6 by setting $u_\alpha(x) = x^\alpha$ for $\alpha_i = 0.5, 1$ and probabilities $h(0.5) = h(1) = 0.5$):

$$\frac{1}{\zeta_1(x) - x} = \frac{0.5}{\zeta_{0.5}(x) - x} \tag{15}$$

$$\zeta'_1(x) + \zeta'_{0.5}(x) = \frac{0.5}{\zeta_{0.5}(x) - x}(\zeta_{0.5}(x) + \zeta_1(x)) \tag{16}$$

We present the equilibrium strategies in figure 1. It is interesting to note that the strategy for each asymmetric risk-averse bidder is, in this example, identical to the case when all his opponents are equally risk-averse (the symmetric bidder case). However, when the valuation is high enough that the risk-neutral opponents would never outbid the risk-averse bidders, the latter increase their bids at a much lower rate as the valuation increases. A similar effect is true for the risk-seeking bidders as well. In fact, *we can prove this observation, for cases of bidders with identical valuation distribution functions.*

For the case of an $(m+1)^{th}$ price auction, it is still a (weakly) dominant strategy to bid truthfully, i.e. use the same strategy as in the case when all the bidders use the same model:

Fact 1 *In the case of an $(m+1)^{th}$ price sealed-bid auction with N participating bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a risk attitude described by utility function $u_{\alpha_i}()$, it is a (weakly) dominant strategy to bid truthfully: $b_i = v_i$.*

This fact also holds for the case when the opponent models are common knowledge, which is examined in the next section.

3.2 Common Knowledge Of The Opponent Models

In this section we examine the same setting as in the previous one, with the difference that the models of all opponents are common knowledge to all participants.

Theorem 2 *Consider the same setting as that of theorem 1, with the difference that the models α_i of all bidders are now common knowledge. Then, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations:*

$$\sum_{\substack{j=1\\j\neq i}}^{N} \zeta'_{\alpha_j}(x)F'_{\alpha_j}(\zeta_{\alpha_j}(x))) \sum_{s\in P_{m-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F_{\alpha_\mu}(\zeta_{\alpha_\mu}(x)) \prod_{\mu\in s}(1-F_{\alpha_\mu}(\zeta_{\alpha_\mu}(x)))\Big) =$$

$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x)-x)}{(u_{\alpha_i}(\zeta_{\alpha_i}(x)-x)-u_{\alpha_i}(0))} \sum_{s\in P_{m-1,N}^{-\{i\}}} \Big(\prod_{\substack{j\notin s\\ j\neq i}} F_{\alpha_j}(\zeta_{\alpha_j}(x)) \prod_{j\in s}(1-F_{\alpha_j}(\zeta_{\alpha_j}(x)))\Big) \tag{17}$$

with boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L$ for all i such that $v_i^L = \min_j\{v_j^L\}$.

Proof. Similar to the proof of theorem 1, we compute that the distribution from which the bid b_j of an opponent with model α_j is drawn has cdf: $Prob[b_j \leq x|\alpha_j] = F_{\alpha_j}(g_{\alpha_j}^{-1}(x))$. Now, bidder i faces $(N-1)$ opponents, which are the agents $\{1,\ldots,N\}-\{i\}$. The distribution of the k^{th} highest bid $B_i^{(k)}$ among the bids of agent i's opponents is $\Phi_k^i(x) = Prob[B_i^{(k)} \leq x]$. This is computed as:

$$\Phi_k^i(x) = \sum_{l=1}^{k} \sum_{s\in P_{l-1,N}^{-\{i\}}} \Big(\prod_{\substack{j\notin s\\ j\neq i}} F_{\alpha_j}(g_{\alpha_j}^{-1}(x)) \prod_{j\in s}(1-F_{\alpha_j}(g_{\alpha_j}^{-1}(x)))\Big) \tag{18}$$

The derivative of this equation is:

$$\frac{d\Phi_k^i(x)}{dx} = \sum_{\substack{j=1\\j\neq i}}^{N} \frac{d(F_{\alpha_j}(g_{\alpha_j}^{-1}(x)))}{dx} \sum_{s\in P_{k-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F_{\alpha_\mu}(g_{\alpha_\mu}^{-1}(x)) \prod_{\mu\in s}(1-F_{\alpha_\mu}(g_{\alpha_\mu}^{-1}(x)))\Big) \tag{19}$$

Using the same reasoning as in theorem 1, we compute the expected utility of bidder i, who places bid b_i, as:

$$\begin{aligned} EU_i(b_i) =& u_{\alpha_i}(0)\big(1-\Phi_m^i(b_i)\big) \qquad (20)\\ &+ u_{\alpha_i}(v_i-b_i)\big(\Phi_m^i(b_i)-\Phi_{m-1}^i(b_i)\big)\\ &+ \int_0^{b_i} u_{\alpha_i}(v_i-\omega)\frac{d}{d\omega}\big(\Phi_{m-1}^i(b_i)\big)d\omega \end{aligned}$$

The bid which maximizes this expected utility, is found by setting: $\frac{dEU_i}{db_i} = 0$. This becomes:

$$(u_{\alpha_i}(v_i-b_i)-u_{\alpha_i}(0))\frac{d}{db_i}\Phi_m(b_i) = u'_{\alpha_i}(v_i-b_i)\big(\Phi_m^i(b_i)-\Phi_{m-1}^i(b_i)\big) \tag{21}$$

This value b_i is equal to $b_i = g_{\alpha_i}(v_i) \Leftrightarrow v_i = g_{\alpha_i}^{-1}(b_i)$, since it maximizes the expected utility $EU_i(b_i)$. Using this substitution, we derive:

$$\sum_{\substack{j=1\\j\neq i}}^{N} \frac{d(F_{\alpha_j}(g_{\alpha_j}^{-1}(b_i)))}{db_i} \sum_{s\in P_{m-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F_{\alpha_\mu}(g_{\alpha_\mu}^{-1}(b_i)) \prod_{\mu\in s}(1-F_{\alpha_\mu}(g_{\alpha_\mu}^{-1}(b_i)))\Big) =$$

$$\frac{u'_{\alpha_i}(g_{\alpha_i}^{-1}(b_i)-b_i)}{(u_{\alpha_i}(g_{\alpha_i}^{-1}(b_i)-b_i)-u_{\alpha_i}(0))} \sum_{s\in P_{m-1,N}^{-\{i\}}} \Big(\prod_{\substack{j\notin s\\ j\neq i}} F_{\alpha_j}(g_{\alpha_j}^{-1}(b_i)) \prod_{j\in s}(1-F_{\alpha_j}(g_{\alpha_j}^{-1}(b_i)))\Big) \tag{22}$$

Defining $\zeta_{\alpha_i}(x) = g_{\alpha_i}^{-1}(x)$, which means that $\zeta()$ are the inverse of $g()$, we derive the system of differential equations 17 for all possible values of $x = b_i$ and for every agent i with model α_i. The boundary conditions come from the fact that a bidder with the lowest possible valuation that any bidder can have $v_i = v_i^L$ will always bid $b_i = v_i^L$. ■

Computing the solution of this system of differential equations as well as the systems characterizing the equilibria of the next section are done in the manner we described in [10].

4 Asymmetric Competitiveness

In this section, we assume that agents have asymmetric competitiveness (spite). In section 4.1 we present the equations that characterize the equilibria when each bidder doesn't know the models of his opponents, while in section 4.2, we present the same analysis when the models of all participants are common knowledge to all participants.

4.1 The Opponent Models Are Not Known

In this section, we assume that each agent has uncertainty not only for the opponents' valuations, but also for how competitive they are. The competitiveness of an agent is characterized by his competition coefficient α_i, which takes values in $[0,1]$, which is drawn from a known probability distribution $h(\alpha)$. We therefore assume that each agent i knows its own valuation v_i and competition coefficient α_i, and also the distributions F and h from which the valuations v_i and competition coefficients α_i of the other agents are drawn. We assume that the number of possible models are λ, meaning that the possible bidder types have competitiveness $\alpha = \alpha_1,\ldots,\alpha_\lambda$.

Theorem 3 *In the case of an m^{th} price sealed-bid auction with N participating risk-neutral bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a competition coefficient α_i, where v_i and α_i are i.i.d. random variables drawn from distributions $F(v)$ and $h(\alpha)$ respectively, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_{\alpha_i}(x) = g_{\alpha_i}^{-1}(x)$ is the solution of the system of differential equations:*

$$\frac{1-\alpha_i m}{N-m}\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(\zeta_\alpha(x))h(\alpha) = \tag{23}$$

$$\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} \zeta'_\alpha(x)F'(\zeta_\alpha(x))\big(\zeta_\alpha(x)-x-\alpha_i\zeta_\alpha(x)+\alpha_i\zeta_\alpha(x)\big)h(\alpha)$$

with boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L, \forall\alpha_i$.

Proof. We assume that the equilibrium strategy is described by functions $g_\alpha(v)$ which map the valuations v to bids for any of the competition factors α. We will use this knowledge to determine the bids of the opponents and the expected profit that a bidder i gets from placing a bid equal to b_i. The distributions of any one opponent bid b_j and of the k^{th} highest opponent bid $B^{(k)}$ are given from equations 7 and 8 (use the same reasoning as in theorem 1). Now, bidder i bids b_i, the bid that maximizes his objective function on expectation.

Let C be the sum (on expectation) of the $(m-1)$ opponent valuations that produced the top $m-1$ winning bids. Since in all cases that we will examine, whether bidder i wins or not, we know that the opponents with the top $(m-1)$ bids will each win an item, we know that they will gain this amount C from doing so. This value is a constant and does not depend on the bid b_i. We will mostly ignore this term in the rest of the computations.

Depending on the bid b_i, we need to consider the following cases: (i) When $B^{(m)} > b_i$, bidder i does not win any item and the closing price is $B^{(m)}$. Therefore bidder i's gain is 0 and the opponents make a gain from gaining an extra item (the m^{th}), in addition to the $(m-1)$ items that they always win (this was counted in the constant value C). We must compute the expected gain obtained by getting this extra item. Let us assume that the actual value of $B^{(m)} = x$. This is equal to a bid submitted by an agent (w.l.o.g. assume this is agent j). Then $b_j = x$ and we want to find out the expectation of the value of the valuation v_j that generated this bid for all possible values of α_j. Let us denote this by $EV(x) = E(v_j|b_j = x)$. For a particular value of α_j, i.e. when $\alpha_j = \alpha$, we know that $v_j = g_\alpha^{-1}(x)$ and this happens with probability $Prob[b_j = x|\alpha_j = \alpha] = \frac{d}{dx}F(g_\alpha^{-1}(x))$. Using Bayes' rule we can compute the value of $E(v_j|b_j = x)$ being equal to:

$$EV(x) = \frac{\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} g_\alpha^{-1}(x)\frac{d}{dx}F(g_\alpha^{-1}(x))h(\alpha)}{\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} \frac{d}{dx}F(g_\alpha^{-1}(x))h(\alpha)} \tag{24}$$

They also must make total payments of $mB^{(m)}$. The total additional[6] expected utility for bidder i in this case is hence:

$$\Delta U_1 = \alpha_i \int_{b_i}^{\infty} \big(m\omega - EV(\omega)\big)\frac{d}{d\omega}\big(\Phi_m(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(\omega))h(\alpha))\big)d\omega \tag{25}$$

(ii) When $B^{(m-1)} > b_i \geq B^{(m)}$, bidder i wins an item and the closing price is b_i. Therefore bidder i's gain is $v_i - b_i$ and the opponents pay $(m-1)b_i$ for the items that they win. The total additional expected utility for bidder i is:

$$\Delta U_2 = \big((1-\alpha_i)(v_i-b_i)+\alpha_i(m-1)b_i\big)\Delta\Phi_m(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)) \tag{26}$$

(iii) When $b_i \geq B^{(m-1)}$, bidder i wins an item and the closing price is $B^{(m-1)}$. Therefore bidder i's gain is $v_i - B^{(m-1)}$ and the opponents must pay $(m-1)B^{(m-1)}$ for the items that they purchase. The total additional expected utility for bidder i in this case is:

$$\Delta U_3 = \int_0^{b_i} \big((1-\alpha_i)(v_i-\omega)+\alpha_i(m-1)\omega\big) \tag{27}$$

$$\frac{d}{d\omega}\big(\Phi_{m-1}(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(\omega))h(\alpha))\big)d\omega$$

The total expected utility for bidder i when considering all possibilities is therefore: $EU_i(b_i) = -\alpha_i C + \Delta U_1 + \Delta U_2 + \Delta U_3$. To find the value of v_i that maximizes the expected utility $EU_i(b_i)$, we set $\frac{dEU_i(b_i)}{db_i} = 0$. We then get:

$$(1-\alpha_i m)\Delta\Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big) = \tag{28}$$

$$\big(v_i - b_i - \alpha_i v_i + \alpha_i EV(b_i)\big)\frac{d}{db_i}\Big(\Phi_m\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha)\big)\Big)$$

By using equation 3 to simplify equation 28, we get:

$$\frac{1-\alpha_i m}{N-m}\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha) = \tag{29}$$

$$\big(v_i - b_i - \alpha_i v_i + \alpha_i EV(b_i)\big)\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}\frac{d}{db_i}F(g_\alpha^{-1}(b_i))h(\alpha)$$

Since strategy $g_\alpha(v)$ gives the equilibrium strategy, then it must be the case that the value of b_i that maximizes the total utility is given by $g_\alpha(v)$, i.e. that $b_i = g_\alpha(v_i) \Leftrightarrow v_i = g_\alpha^{-1}(b_i)$. Using this fact and equation 24 to substitute in equation 29, we get:

$$\frac{1-\alpha_i m}{N-m}\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(g_\alpha^{-1}(b_i))h(\alpha) = \tag{30}$$

$$\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}\frac{F'(g_\alpha^{-1}(b_i))}{g'_\alpha(g_\alpha^{-1}(b_i))}\big(g_\alpha^{-1}(b_i) - b_i - \alpha_i g_\alpha^{-1}(b_i) + \alpha_i g_\alpha^{-1}(b_i)\big)h(\alpha)$$

Setting $\zeta_{\alpha_i}(x) = g_{\alpha_i}^{-1}(x)$, we derive the system of differential equations 23 for all possible values of $x = b_i$ and for every agent i with model α_i. We select the boundary condition $g_\alpha(v_l) = v_l$, based on the fact that this boundary condition holds when all the agents have the same competition factor α, $\forall\alpha$. (see [9]) ■

Theorem 4 *In the case of an $(m+1)^{th}$ price sealed-bid auction with N participating risk-neutral bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a competition coefficient α_i, where v_i and α_i are i.i.d. random variables drawn from distributions $F(v)$ and $h(\alpha)$ respectively, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_{\alpha_i}(x) = g_{\alpha_i}^{-1}(x)$ is the solution of the system of differential equations:*

$$\forall v_i, \alpha_i : -\alpha_i\Big(1 - \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F(\zeta_\alpha(x))h(\alpha)\Big) = \tag{31}$$

$$\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} \zeta'_\alpha(x)F'(\zeta_\alpha(x))\big(g_\alpha^{-1}(x) - x - \alpha_i g_\alpha^{-1}(x) + \alpha_i\zeta_\alpha(x)\big)h(\alpha)$$

with boundary conditions: $g_{\alpha_i}(v_i^H) = v_i^H, \forall\alpha_i$.

Proof. This proof as well as the proof of theorem 6 are omitted due to space. ■

4.2 Common Knowledge Of The Opponent Models

In this section we examine the same setting as in the previous one, with the difference that the models of all opponents are common knowledge. We initially examine the setting where an m^{th} price auction is used:

Theorem 5 *Consider the same setting as that of theorem 3, with the difference that the models α_i of all bidders are now common knowledge. Then, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations:*

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \zeta'_{\alpha_j}(x)F'(\zeta_{\alpha_j}(x)))(\zeta_{\alpha_i}(x) - x - \alpha_i\zeta_{\alpha_i}(x) + \alpha_i\zeta_{\alpha_j}(x))\cdot$$

$$\sum_{s\in P_{m-1,N}^{-\{i,j\}}}\Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F(\zeta_{\alpha_\mu}(x))\prod_{\mu\in s}(1 - F(\zeta_{\alpha_\mu}(x)))\Big) =$$

$$(1-\alpha_i m)\sum_{s\in P_{m-1,N}^{-\{i\}}}\Big(\prod_{\substack{j\notin s\\ j\neq i}} F(\zeta_{\alpha_j}(x))\prod_{j\in s}(1 - F(\zeta_{\alpha_j}(x)))\Big) \tag{32}$$

with boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L, \forall\alpha_i$.

Proof. Similar to the proof of theorem 3, we compute that the distribution from which the bid b_j of an opponent with model α_j is drawn has cdf: $Prob[b_j \leq x|\alpha_j] = F(g_{\alpha_j}^{-1}(x))$. Now, bidder i faces $(N-1)$ opponents, which are the agents $\{1,\ldots,N\} - \{i\}$. The distribution of the k^{th} highest bid $B_i^{(k)}$ among the bids of agent i's

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[6] We mean additional to the fact that the agent always loses utility αC, since its opponents always gain a value C from the top $(m-1)$ items.

opponents is $\Phi_k^i(x) = Prob[B_i^{(k)} \leq x]$, which is given by equation 18, and the pdf of this distribution by equation 19. We compute the expected utility of bidder i, who places bid b_i, as:
$EU_i(b_i) = -\alpha_i C + \Delta U_1 + \Delta U_2 + \Delta U_3$, where:

$$\Delta U_1 = \alpha_i \int_{b_i}^{\infty} \Big(m\omega \frac{d}{d\omega} \Phi_m^i(\omega) - \tag{33}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} g_{\alpha_j}^{-1}(\omega) \frac{d(F(g_{\alpha_j}^{-1}(\omega)))}{d\omega} \sum_{s\in P_{m-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F(g_{\alpha_\mu}^{-1}(\omega)) \prod_{\mu\in s} (1 - F(g_{\alpha_\mu}^{-1}(\omega)))\Big)\Big) d\omega$$

$$\Delta U_2 = \big((1-\alpha_i)(v_i - b_i) + \alpha_i(m-1)b_i\big)\big(\Phi_m^i(b_i) - \Phi_{m-1}^i(b_i)\big) \tag{34}$$

$$\Delta U_3 = \int_0^{b_i} \big((1-\alpha_i)(v_i - \omega) + \alpha_i(m-1)\omega\big) \frac{d}{d\omega} \Phi_{m-1}^i(\omega) d\omega \tag{35}$$

In order to compute equations 34 and 35, we used the same reasoning as in theorem 3; these cover that cases that the bid b_i of bidder i is respectively the m^{th} highest bid and higher than the m^{th} bid, and therefore, in both cases, the bidder wins an item. Equation 33 represents the case that b_i is less than the m^{th} highest opponent bid in which case ΔU_1 is α_i times the difference of the expected payments made by the opponents minus the expected gain of the opponent who placed the m^{th} highest bid from getting the item; the first term is computed as in theorem 3. The second term is computed by not only examining the probability of the m^{th} highest opponent bid being equal to $\omega < b_i$, but also conditional on the fact that this bid was placed by a particular opponent j; we account for all possible opponents j and thus derive the second term of equation 33.

To find the value of v_i that maximizes the expected utility $EU_i(b_i)$, we set $\frac{dEU_i(b_i)}{db_i} = 0$. We then get:

$$(v_i - b_i - \alpha_i v_i) \frac{d}{db_i} \Phi_m^i(b_i) + \alpha_i \cdot$$

$$\sum_{\substack{j=1\\ j\neq i}}^{N} g_{\alpha_j}^{-1}(b_i) \frac{d(F(g_{\alpha_j}^{-1}(b_i)))}{db_i} \sum_{s\in P_{m-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F(g_{\alpha_\mu}^{-1}(b_i)) \prod_{\mu\in s} (1 - F(g_{\alpha_\mu}^{-1}(b_i)))\Big)\Big) db_i$$

$$= (1 - \alpha_i m)(\Phi_m^i(b_i) - \Phi_{m-1}^i(b_i))$$

This value b_i is equal to $b_i = g_{\alpha_i}(v_i) \Leftrightarrow v_i = g_{\alpha_i}^{-1}(b_i)$, since it maximizes the expected utility $EU_i(b_i)$. Using this substitution and using equations 18 and 19, we derive:

$$\sum_{\substack{j=1\\ j\neq i}}^{N} (g_{\alpha_i}^{-1}(b_i) - b_i - \alpha_i g_{\alpha_i}^{-1}(b_i) + \alpha_i g_{\alpha_j}^{-1}(b_i)) \frac{d(F(g_{\alpha_j}^{-1}(b_i)))}{db_i} \cdot$$

$$\sum_{s\in P_{m-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F(g_{\alpha_\mu}^{-1}(b_i)) \prod_{\mu\in s} (1 - F(g_{\alpha_\mu}^{-1}(b_i)))\Big)\Big) db_i$$

$$= (1 - \alpha_i m) \sum_{s\in P_{m-1,N}^{-\{i\}}} \Big(\prod_{\substack{j\notin s\\ j\neq i}} F(g_{\alpha_j}^{-1}(b_i)) \prod_{j\in s} (1 - F(g_{\alpha_j}^{-1}(b_i)))\Big)$$

Defining $\zeta_{\alpha_i}(x) = g_{\alpha_i}^{-1}(x)$, which means that $\zeta()$ are the inverse of $g()$, we derive the system of differential equations 32 for all possible values of $x = b_i$ and for every agent i with model α_i. The boundary condition is $g_{\alpha_i}(v_i^L) = v_i^L$, as in the case when the opponent models are not known. ■

Theorem 6 *Consider the same setting as that of theorem 4, with the difference that the models α_i of all bidders are now common knowledge. Then, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations:*

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \zeta'_{\alpha_j}(x) F'(\zeta_{\alpha_j}(x))(\zeta_{\alpha_i}(x) - x - \alpha_i \zeta_{\alpha_i}(x) + \alpha_i \zeta_{\alpha_j}(x)) \cdot$$

$$\sum_{s\in P_{m-1,N}^{-\{i,j\}}} \Big(\prod_{\substack{\mu\notin s\\ \mu\neq i,j}} F(\zeta_{\alpha_\mu}(x)) \prod_{\mu\in s} (1 - F(\zeta_{\alpha_\mu}(x)))\Big) =$$

$$-\alpha_i m \sum_{s\in P_{m,N}^{-\{i\}}} \Big(\prod_{\substack{j\notin s\\ j\neq i}} F(\zeta_{\alpha_j}(x)) \prod_{j\in s} (1 - F(\zeta_{\alpha_j}(x)))\Big) \tag{36}$$

with boundary conditions: $g_{\alpha_i}(v_i^H) = v_i^H, \forall \alpha_i$.

5 Conclusions

In this paper, we examined asymmetric bidder models both in risk attitudes, valuations and competitiveness. We gave the systems of differential equations that characterize the Bayes-Nash equilibria in these cases, both when the models of all bidders are common knowledge as well as when there is uncertainty about the models of the opponents. We examined both settings where m^{th} and $(m+1)^{th}$ price auctions are being used.

There are still a number of issues we are currently pursuing. The foremost of these is that we are exploring different methods of solving the systems of differential equations which characterize the equilibria of this paper. In this paper, we did not present how to solve these equations, other than the example of section 3.1. Our research has concluded that these particular systems of differential equations are inherently unstable around the point specified by the boundary conditions. Therefore, when using methods based on the Taylor expansion, such as the Runge-Kutta variant we presented in [10], which can solve all these systems, it is not guaranteed that a solution will be found, even if one exists, exactly due to this instability. To this end, we are currently working also with symbolic solutions to these equations, which give the solutions of the systems presented in this paper, but are less general than the equivalent numerical methods. This will allow us to examine the cases when these solutions exist. Furthermore, it is clear that being able to analyze auctions with asymmetric bidders will facilitate the analysis of a number of real world scenarios; for example, we are examining the application of our results to service procuring scenarios.

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