Sequential Auctions With Partially Substitutable Goods

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Abstract. In this paper, we examine a setting in which a number of partially substitutable goods is sold in sequential single unit auctions. Each bidder needs to buy exactly one of these goods. In previous work, this setting has been simplified by assuming that bidders do not know their valuations for all items a priori, but rather are informed of their true valuation for each item right before the corresponding auction takes place. In our analysis we don't make this assumption. This complicates significantly the computation of the equilibrium strategies. We examine this setting both for first and second-price auction variants, initially when the closing prices are not announced, for which case we prove that sequential first and second-price auctions are revenue equivalent, and then when the prices are announced; in the latter case, because of the asymmetry in the announced prices between the two auction variants, revenue equivalence does not hold. We finish the paper, by giving some initial results about the case when free disposal is allowed, and therefore a bidder can purchase more than one item.

1 Introduction

Auctions have become commonplace when allocating resources in multiagent systems; they are used to trade all kinds of commodity, from flowers and food to keyword targeted advertisement slots, from bonds and securities to spectrum rights. There are several auction formats that can be used when selling a group of items; some of the best known ones are combinatorial auctions [1], parallel auctions [2], or sequential auctions [3, 4]. Out of these, separate (i.e. non combinatorial) auctions each selling a single item are the most common case on the internet, since they are easy to implement. In most interesting scenarios these auctions are analyzed as sequential auctions, because their closing times do not coincide and therefore the result of the earlier ones are known by the time that a bidder would bid for the later ones.

Using sequential auctions to sell a number of items is the model that we examine in this paper. In [4], the authors examined a model where an auctioneer sells a number of partially substitutable items to a number of bidders. Their motivating example is that of advertisers bidding for advertising space on a particular network. There are several possible advertisement slots available, each associated with a particular program and time, and each advertiser wants to advertise on this network only once. The slots are therefore partially substitutable. However, different slots have a different value to each advertiser depending on the audience towards which he would like to target his ad. Another motivating example is that of a person who wants to buy a painting for her office; there is space for one painting only and there are several paintings to be auctioned off at a particular auction house, where this person visits in order to purchase a painting. The different paintings have different values not only because of their inherent valuation, but also because of the personal preferences of the buyer.

Now, in [4], the authors assumed that the valuations of the different items do not become known to the bidders until after all the previous auctions have been concluded. This simplifies the analysis of how bidders bid, because in previous auctions they all have the same expected profit from the remaining auctions, given that they don't each know how much they value the remaining items. It is more realistic to remove this assumption and let each bidder know his entire set of valuations from the beginning of the game. This means that the bid placed at each auction depends on the entire vector of valuations rather than a single one, which was the case when the valuations are not known a priori. Therefore, in this paper we focus on computing the equilibrium strategies of the bidders, while in [4], the focus was on computing the optimal agenda (i.e. order of the items being auctioned) of the seller in order to maximize her revenue. The other difference is that here we examine both first and second price auctions, whereas in [4] only sequential second price auctions were considered.

This paper is organized as follows. In section 2, we formally present the setting that we will analyze. In section 3, we give the equilibrium strategies for the sequential first price and second price auctions with the assumption that the closing prices are not announced; under this setting we prove that the two auction variants are revenue equivalent. In section 4, we include in our analysis also the information obtained from announcements about the closing prices of the auctions; the equilibrium strategies computed in this section are generalizations of those in the previous section. In section 5, we proceed to give a small example to illustrate and clarify how the computation of the equilibria is executed both when prices are not announced and when they are. In section 6, we discuss about other related work and conclude. Finally, in the appendix, we give some initial results of the case when free disposal is allowed.

2 Problem Setup

In this section we formally describe the auction setting to be analyzed and give the notation that we use. The setting is similar to that in [4] with the exception that each bidder knows all his valuations a priori.

In particular, there are $m > 1$ sequential auctions each selling a single item. The order in which these goods are sold is common knowledge. There are $n > 1$ bidders participating in these auctions. Each bidder i has a different independent valuation v_j^i for the good sold in the j^{th} auction. In general, these valuations are i.i.d. variables and the valuations of bidders for the j^{th} item are drawn from known distributions with cumulative density function (cdf) $F_j(), \forall j \leq m$. These goods are partially substitutable in the sense that each bidder can buy only one good and it can use (and bid for) any one of them. The bidders must bid for all items, although they have the option of placing bids equal to 0, in which case they might win only if all bidders place bids equal to zero. In such a case of a tie, the winner is selected randomly. Once a bidder has won any item it stops bidding in the remaining auctions. The bidders are risk neutral and only care about maximizing their utility, which is defined as the difference between the valuation v_j^i minus the price paid, if they win the j^{th} auction, or 0 if they don't win any auctions.

We will compute and analyze the symmetric Bayes-Nash equilibria that exist in sequential first auctions and sequential second price auctions. The equilibria that we compute are defined by a bidding strategy, which maps the agents' valuations v_j^i to bids b_j^i. There are a number of other parameters that we will use in order to compute these bids, but we define those at the corresponding theorems. In the case that the closing prices are announced, then the bidding strategies depend on these parameters as well, and we will describe how to incorporate this information in our analysis.

3 Equilibrium Computation When No Prices Are Announced

In this section, we examine the strategic behavior of bidders when the closing prices of previous auctions are not announced and therefore the bidders do not learn any information about the private values of their opponents.

Theorem 1. *Assume that n bidders participate in m sequential second-price auctions, each selling one item. The items are assumed to be partially substitutable and each bidder is interested in purchasing exactly one item. The valuation of the item sold in the j^{th} auction to bidder i is v_j^i. The valuations for the j^{th} item to the bidders are i.i.d. variables drawn from prior distribution $F_j(v)$. The valuations that each bidder has for different items are independent. In this scenario, it is a symmetric Bayes-Nash equilibrium strategy*[1] *for each agent to bid:*

$$b_j^i = \max\left\{0, v_j^i - \sum_{k=j+1}^{m} EP_k^i\right\} \tag{1}$$

where EP_k^i, the expected profit of bidder i from the k^{th} auction, is:

$$EP_j^i = \begin{cases} \int_{0^-}^{b_j^i} \left(v_j^i - \omega - \sum_{k=j+1}^{m} EP_k^i\right) \frac{d}{d\omega} \Phi_j^{n-j}(\omega) d\omega & b_j^i > 0 \\ \frac{1}{n-j+1}\left(v_j^i - \sum_{k=j+1}^{m} EP_k^i\right) \Phi_j^{n-j}(0) & b_j^i = 0 \end{cases} \tag{2}$$

and $\Phi_j()$, the pdf of any opponent bid in the j^{th} auction, is:

$$\Phi_j(x) = \underbrace{\int_0^\infty \dots \int_0^\infty}_{(m-j)\ integrals} F_j(x + \sum_{k=j+1}^{m} EP_k^i) \cdot \underbrace{F'_{j+1}(\omega_{j+1}) \dots F'_m(\omega_m)}_{(m-j)\ terms} \cdot \underbrace{d\omega_{j+1} \dots d\omega_m}_{(m-j)\ vars} \tag{3}$$

[1] To be precise at each auction it is a weakly dominant strategy for the bidders to bid in this way, provided that other bidders will play the dominant strategy in the remaining rounds. Because of this dependence on bidders playing the dominant strategy in the remaining rounds, in order to have the computed strategy be the dominant strategy in the current round (auction), the overall solution concept is a Nash equilibrium.

Proof. Proof by induction. In the last (m^{th}) auction, there are $(N-m+1)$ participating bidders, as $(m-1)$ of the other bidders have won an item and have dropped out, and we know that it is a weakly dominant strategy in this case to bid truthfully. Thus the equations hold when $j = m$.

Assume now that the equations are correct for all auctions between the $(j+1)^{th}$ and m^{th}. We now need to prove that they hold for the j^{th} auction. From the point of view of each bidder i, it faces $(N-j)$ opponents, as $(j-1)$ opponents have already won one item each and left. The expected profit of bidder i when he bids b_j^i in the j^{th} auction is:

$$EP_j^i = \int_{0^-}^{b_j^i} \big(v_j^i - \omega - \sum_{k=j+1}^{m} EP_k^i\big) \frac{d}{d\omega} \Phi_j^{n-j}(\omega) d\omega \tag{4}$$

because if he wins he will get profit equal to $v_j^i - \omega$, where ω is the second highest bid (the highest opponent bid), and he will lose the profit $\sum_{k=j+1}^{m} EP_k^i$ that this agent would have made in the later rounds if he were to participate in them (which will not happen because the agent will win and withdraw); the probability of the highest opponent bid being equal to ω has pdf $\Phi_j^{n-j}(\omega)$. The bid which maximizes the expected utility does not actually depend on the opponent bids as described by pdf $\Phi_j()$, and is equal to $b_j^i = v_j^i - \sum_{k=j+1}^{m} EP_k^i$, if this term is positive. Otherwise the bidder expects to gain more from the later round and would not want to bid anything more than $b_j^i = 0$. Thus we get equation 1.

From equation 4, we get equation 2, for the case when $b_j^i > 0$. However, we also need to consider the case when $b_j^i = 0$. In this case the bidder will win only if all the other bidders will also bid 0, which happens with probability equal to $\Phi_j^{n-j}(0)$; one bidder will be selected randomly with (equal for each bidder) probability $\frac{1}{n-j+1}$ and in this instance the bidder will make profit $v_j^i - \sum_{k=j+1}^{m} EP_k^i \leq 0$.

To get equation 3, we must consider all possible cases for the values that can be obtained by the valuations for the items in auctions $(j+1)$ through m for any opponent i; we denote these values by $\omega_{j+1}, \ldots, \omega_m$. The probability of this case happening is equal to $F_{j+1}^{'}(\omega_{j+1}) \ldots F_m^{'}(\omega_m)$. Given these values for all the valuations in the later auctions we can now compute the expected profit in all the later auctions $EP_k^i, \forall k = j+1, \ldots, m$. From equation 1 we know that $b_j^i = \max\big\{0, v_j^i - \sum_{k=j+1}^{m} EP_k^i\big\}$. Thus, the probability that $b_j^i = 0$ is equal to the probability that $v_j^i \leq \sum_{k=j+1}^{m} EP_k^i$, which is $F(\sum_{k=j+1}^{m} EP_k^i)$. Additionally, the probability that $b_j^i \leq x$ (for $x > 0$) is equal to the probability that $v_j^i - \sum_{k=j+1}^{m} EP_k^i \leq x \Leftrightarrow v_j^i \leq x + \sum_{k=j+1}^{m} EP_k^i$, which is $F(x + \sum_{k=j+1}^{m} EP_k^i)$. So $\forall x \geq 0$ the probability that $b_j^i \leq x$ is $F(x + \sum_{k=j+1}^{m} EP_k^i)$. All these observations give us equation 3.

Thus we were able to prove that all the equations hold for the j^{th} auction, which completes the proof by induction.

Note that the crucial parameter in each auction is the valuation for the item discounted by the expected profit in the later auctions: $v_j^i - \sum_{k=j+1}^{m} EP_k^i$, which is the additional value they expect to gain from this item; we'll call this term the "discounted valuation" from now on. The agents bid truthfully in the sense that they bid this discounted valuation (if it is positive). We will show in the next theorem that this value is the crucial parameter also in the case of sequential first-price auctions.

Theorem 2. *Assume the same setting as theorem 1 with the difference that the auctions are now first price auctions. In this scenario, it is a symmetric Bayes-Nash equilibrium strategy for each agent to bid:*

$$b_j^i = g_j\Big(\max\big\{0, v_j^i - \sum_{k=j+1}^{m} EP_k^i\big\}\Big) \tag{5}$$

where $\Phi_j()$, the probability distribution of the discounted valuations, is given by equation 3 and

$$EP_j^i = \begin{cases} \big(v_j^i - b_j^i - \sum_{k=j+1}^{m} EP_k^i\big)\Phi_j^{n-j}(g_j^{-1}(b_j^i)) & b_j^i > 0 \\ \frac{1}{n-j+1}\big(v_j^i - \sum_{k=j+1}^{m} EP_k^i\big)\Phi_j^{n-j}(0) & b_j^i = 0 \end{cases} \tag{6}$$

$$g_j(x) = x - \frac{1}{\Phi_j^{n-j}(x)} \int_{0^+}^{x} \Phi_j^{n-j}(\omega) d\omega \tag{7}$$

Proof. Again we will use induction to prove this theorem. In the last (m^{th}) auction, there are $(N - m + 1)$ participating bidders, and we know that the symmetric Bayes-Nash equilibrium strategy in this case is to bid $b_m^i = v_m^i - \frac{1}{F_m^{n-m}(v_m^i)} \int_0^{v_m^i} F_m^{n-m}(\omega) d\omega$. [5] From this fact it follows that the equations hold when $j = m$.

Assume now that the equations are correct for all auctions between the $(j+1)^{th}$ and m^{th}. We now need to prove that they hold for the j^{th} auction. From the point of view of each bidder i, it faces $(n - j)$ opponents in this auction. The expected profit of bidder i when he bids b_j^i in the j^{th} auction is:

$$EP_j^i = \big(v_j^i - b_j^i - \sum_{k=j+1}^{m} EP_k^i\big)\Phi_j^{n-j}(g_j^{-1}(b_j^i)) \tag{8}$$

because if he wins he will get profit equal to $v_j^i - b_j^i$, and he will lose the profit $\sum_{k=j+1}^{m} EP_k^i$ that this agent would have made in the later rounds by participating in them. The probability of winning when bidding b_j^i is $\Phi_j^{n-j}(g_j^{-1}(b_j^i))$, because the bid b_j^i must be higher than all $(n - j)$ opponent bids and each opponent bid has pdf $\Phi_j(g_j^{-1}(x))$, since $\Phi_j(x)$ is the pdf of the opponent discounted valuations and the bids at the equilibrium are mapped from these discounted valuations by function $g_j()$. The bid which maximizes the expected utility is found by setting $\frac{d}{db_j^i} EP_j^i = 0$. This gives:

$$\Phi_j(g_j^{-1}(x)) = (n - j)\big(v_j^i - b_j^i - \sum_{k=j+1}^{m} EP_k^i\big)\frac{\Phi_j'(g_j^{-1}(b_j^i))}{g_j'(g_j^{-1}(b_j^i))} \tag{9}$$

Given that, at the equilibrium, it must be $b_j^i = g_j\big(v_j^i - \sum_{k=j+1}^{m} EP_k^i\big)$, when the discounted valuation $v_j^i - \sum_{k=j+1}^{m} EP_k^i$ is positive (or 0 otherwise, when the discounted valuation is zero or negative), and the fact that the boundary condition is

$\lim_{x \to 0^+} g_j(x) = 0$, it follows that the solution of differential equation 9 is indeed given by equation 7.

Given that the discounted valuation $v^i_j - \sum_{k=j+1}^m EP^i_k$ is the crucial parameter in both first and second price auction settings, we can now prove that the two settings are revenue equivalent:

Corollary 1. *The expected revenue of the seller and the bidders is (in expectation) the same in both the case when m sequential first-price auctions are used and the case when m sequential second-price auctions are used.*

Proof. We give a short proof. It is sufficient to show that the expected revenue EP^i_k of bidder i in the k^{th} auction is the same under both settings. We do this by using induction. In the last auction, the two auctions (first and second-price) are known to be revenue equivalent; see [5]. Assume that EP^i_k is the same in all the auctions between the $(j+1)^{th}$ and m^{th}. Now since the discounted valuations for the j^{th} auction are given by the same formula $v^i_j - \sum_{k=j+1}^m EP^i_k$ for both settings and the expected profits EP^i_k are the same $\forall i = j+1, \ldots, m$ for both settings, it follows that the discounted valuations are the same for both settings and are given by the same pdf $\Phi_j()$. Given that, according to equation 7, in the sequential first-price auctions setting, the discounted valuation $v^i_j - \sum_{k=j+1}^m EP^i_k$ of each bidder i is mapped to a bid equal to the expected value of the highest opponent discounted value, conditional on the fact that this highest opponent discounted value is lower than $v^i_j - \sum_{k=j+1}^m EP^i_k$, and that, in the sequential second-price auctions setting, the bidders bid truthfully their discounted valuations, it follows that in both settings the expected seller profit is equal to the expected value of the second highest, among all bidders, discounted valuation. From this it also follows that the expected profits of the bidders EP^i_j in the j^{th} auction are the same under both auction settings.

4 Equilibrium Computation When The Prices Are Announced

In this section, we extend the results of the previous section to include knowledge of the prices p_i ($i < j$) paid by winning bidders in the first $(j-1)$ auctions, when the bidding strategy in the j^{th} auction is considered. These prices can be mapped to the discounted valuations that produced them and this gives some knowledge of the discounted valuations of the opponents that remain in the auction. In the first price setting, the information learned is that all remaining bidders have discounted valuations for the previous rounds which are smaller or equal than the respective discounted valuations of the winning bidders. In the second price setting, some similar information is learned, meaning that all discounted valuations are smaller than the discounted valuations that correspond to the announced prices, with one exception: as each announced price corresponds to the second highest bid, this means that one of the remaining bidders has a discounted valuation with is equal (and not potentially smaller) than the valuation which is mapped from the announced price. Now, in the case when all the items are the same, which is examined in [5], this poses no problem for the analysis, because this bidder will win the auction that immediately follows and the fact that the remaining bidders have learned his private valuation does not matter as it is a dominant strategy for them to discount their valuation according to the expected gain in future auctions (in which this bidder will no longer participate). However, in the setting that we examine, there is no guarantee that this bidder will win the next auction and thus his valuation will potentially influence the bidding of the remaining bidders. Furthermore, it is entirely likely that different bidders, which had set the prices for some previous auctions, will remain in the future auctions. In order to be 100% accurate in our analysis of the second price setting, we would have to examine all possible cases for which bidder had had the second highest bid and thus had set the price in the previous auctions, what is the probability that this bidder remains in each future auction, and also what is probability that the same bidder has set the price in more than one previous auctions (and which auctions specifically). We will instead make the assumption that all the discounted valuations are smaller or equal to one that corresponds to each announced price and thus ignore the fact that we know that one of these (but not exactly which one) is equal to the closing price.[2] It might initially seem that this assumption could impact significantly the bids of the remaining bidders. However, for an agent to have such a high valuation it would mean that both his valuation for the auction that just closed was relatively high and that his valuations for subsequent auctions are lower than most of the other participants. Therefore, it is unlikely that this bidder will have the highest discounted valuation in any of the remaining auctions. By assuming that his discounted valuation can be smaller we do increase somewhat the computed probability of him having a higher discounted valuation in future auctions, but this makes small difference to the final computations, especially in cases where the total number of bidders is significantly higher than the number of items sold.

On the other hand *if the identity of the second highest bidder is revealed together with the bid, we need not make the previous assumption*, but we can rather incorporate the fact that we know his discounted valuation precisely into our analysis. While the next theorem is presented with this assumption we will discuss afterwards how it is modified in order to account for this knowledge in the case that the identities of the second highest bidders are also announced. In the theorems that follow we will use the following function:

Definition 1. *The indicator function $\mathcal{I}$ takes as input a boolean expression e and returns:*
$\mathcal{I}(e) = 1$*, if expression* $e = true$*, and*
$\mathcal{I}(e) = 0$*, if expression* $e = false$*.*

Theorem 3. *Assume the same setting as theorem 1 with the difference that the prices $\overrightarrow{p} = \{p_k\}(k = 1, \ldots, j-1)$, at which the previous auctions have closed, are announced and are common knowledge to bidders at the j^{th} auction. In this scenario, it is a symmetric Bayes-Nash equilibrium strategy for each agent to bid:*

[2] We only need make this assumption, when the identity of the bidder who set the price is not known. As we explain later in this section, if this information is available, this assumption is not made, and thus the equilibrium strategy is computed entirely accurately.

$$b_j^i = \max\left\{0, v_j^i - \sum_{k=j+1}^{m} EP_{j,k}^i\right\} \tag{10}$$

where $EP_{j,k}^i$, the expected profit of bidder i from the k^{th} auction, as computed during the j^{th} auction[3]*, is:*

$$EP_{j,k}^i = \begin{cases} \int_{0^-}^{b_k^i} \left(v_k^i - \omega - \sum_{\lambda=k+1}^{m} EP_{j,\lambda}^i\right) \frac{d}{d\omega} \Phi_{j,k}^{n-k}(\omega) d\omega & b_k^i > 0 \\ \frac{1}{n-k+1}\left(v_k^i - \sum_{\lambda=k+1}^{m} EP_{j,\lambda}^i\right) \Phi_{j,k}^{n-k}(0) & b_k^i = 0 \end{cases} \tag{11}$$

and $\Phi_{j,k}()$, the pdf of any opponent bid in the k^{th} auction, as computed during the j^{th} auction, is given by dividing $N(x, \overrightarrow{p})$, the probability that any combination of valuations gives discounted valuations which are consistent with the prices $\overrightarrow{p}$ observed in the previous auctions and that the bid in the current auction would be $\leq x$, by $D(\overrightarrow{p})$, the probability that any combination of valuations gives discounted valuations which are consistent with the prices $\overrightarrow{p}$ observed in the previous auctions. Thus, $\Phi_{j,k}() = \frac{N(x,\overrightarrow{p})}{D(\overrightarrow{p})}$. These terms are computed as follows:

$$N(x, \overrightarrow{p}) = \int_0^{\infty} \ldots \int_0^{\infty} F_1^{'}(\omega_1) \ldots F_m^{'}(\omega_m) \cdot d\omega_1 \ldots d\omega_m \cdot \mathcal{I}(\{\omega_j - \sum_{k=j+1}^{m} EP_{j,k}^i \leq x\} \wedge \bigwedge_{\lambda<j} \{\omega_\lambda - \sum_{k=\lambda+1}^{m} EP_{\lambda,k}^i \leq p_\lambda\}) \tag{12}$$

$$D(\overrightarrow{p}) = \int_0^{\infty} \ldots \int_0^{\infty} \mathcal{I}(\bigwedge_{\lambda<j} \{\omega_\lambda - \sum_{k=\lambda+1}^{m} EP_{\lambda,k}^i \leq p_\lambda\}) \cdot F_1^{'}(\omega_1) \ldots F_m^{'}(\omega_m) \cdot d\omega_1 \ldots d\omega_m \tag{13}$$

Proof. Again we do this proof by induction. However, we are going to need a double induction both on j, the number of the current auction at which the computation is performed, and on k, which is number of the auction examined.

The first induction is on j. When $j = 1$ there is no price information and, we can verify easily that the equations of this theorem generate the equations of theorem 1. Now assume that the theorem is correct when the computation is performed before auctions $1, \ldots, j-1$. We have computed both the bidding strategies and the expected profit at all rounds before each auction started. We need to show that the equations hold also when we perform the same computations before the j^{th} auction is run.

To prove this fact, we need a second induction, which is fairly similar to the proof of theorem 1. We need the equations to hold for any auction $k = j, \ldots, m$ considered. In the last (m^{th}) auction, there are $(N - m + 1)$ participating bidders, as $(m-1)$ of the other bidders have won an item and have dropped out, and we know that it is a weakly dominant strategy in this case to bid truthfully. Thus the equations hold when $k = m$.

[3] At that stage, right before the j^{th} auction has been contacted, the closing prices $p_1, \ldots, p_{j-1}$ have been observed. These observations change the expected profit from future rounds and this is the reason why we need to index the expected future profits by when the round at which this computations is made.

Assume now that the equations are correct for all auctions between the $(k+1)^{th}$ and m^{th}. We now need to prove that they hold for the k^{th} auction. From the point of view of each bidder i, it faces $(N-k)$ opponents, as $(k-1)$ opponents have already won one item each and left. Most of the proof now is similar to that of theorem 1. What is different essentially is the computation of the distribution of any opponent's bid, because the closing prices $\overrightarrow{p}$ must be included in the agents' reasoning.

To compute $\Phi_{j,k}()$ we need to prove the validity of equations 12 and 13. We must consider all possible values that the valuations of any opponent i can have for the items in auctions 1 through m; we denote these values by $\omega_1, \ldots, \omega_m$. The probability of this particular combination happening is equal to $F_1^{'}(\omega_1) \ldots F_m^{'}(\omega_m)$. Given these values for all the valuations that the opponent might have in all auctions, we need to compute the conditional probability of him having a discounted valuation less or equal to x, given feasible combinations of valuations. Now, we know from the announced prices that the bid (which was equal to the discounted valuation) of that opponent in the λ^{th} auction (as computed right before the λ^{th} auction) was:

$$\omega_\lambda - \sum_{k=\lambda+1}^{m} EP^i_{\lambda,k} \leq p_\lambda$$

If valuations $v_1 = \omega_1, \ldots, v_m = \omega_m$ do not satisfy all these inequalities then that combination of valuations is not supported by the observed closing prices and is not feasible. Thus equation 13 is correct because it gives the probability that the combination of valuations is feasible. Similarly, the nominator given by equation 12 is all those combinations which, in addition, give a discounted valuation (which is equal to the bid) smaller or equal to x. We know that by dividing the two terms we get the desired conditional probability.

This completes the inner inductive step. As a result the theorem correctly computes the bids when making the computations before the j^{th} auction. This is why the outer induction also holds, which completes the proof.

Now, there is a way to extend this theorem to account for knowledge of a particular bidders bids being in some auctions equal to the closing price. Assume that the identities of the bidders who have set the closing prices by placing the second highest bids are announced. Then, for any bidder i that did not set any closing price, the computation of $\Phi^{(i)}(x)$, his probability of bidding up to x in the current auction, remains the same as in theorem 3. However, for the other bidders this is not the case. If it is known that bidder i has set the prices in all auctions $z \in Z$, where $Z \subseteq \{1, \ldots, j-1\}$, then $\omega_z - \sum_{k=z+1}^{m} EP^i_{z,k} = p_z$, for those values $z \in Z$. We need to modify the expressions for $N()$ and $D()$ appropriately. Let us call the new functions $\widetilde{N}()$ and $\widetilde{D}()$. As $\omega_z - \sum_{k=z+1}^{m} EP^i_{z,k} = p_z$ is equivalent to $p_z \leq \omega_z - \sum_{k=z+1}^{m} EP^i_{z,k} \leq p_z + \Delta p_z$, when $\Delta p_z \to 0$, we can easily verify that

$$\widetilde{N}(x, \overrightarrow{p}, z) = \frac{\vartheta^{|Z|} N(x, \overrightarrow{p})}{\vartheta z_1 \ldots \vartheta z_{|Z|}} \Delta z_1 \ldots \Delta z_{|Z|}$$

where $z_1, \ldots, \Delta z_{|Z|}$ are the elements of set Z. Similarly:

$$\widetilde{D}(\overrightarrow{p}, z) = \frac{\vartheta^{|Z|} D(\overrightarrow{p})}{\vartheta z_1 \dots \vartheta z_{|Z|}} \Delta z_1 \dots \Delta z_{|Z|}$$

and the distribution of this opponent's bid is $\Phi^{(i)}(x) = \frac{\widetilde{N}(x, \overrightarrow{p}, z)}{\widetilde{D}(\overrightarrow{p}, z)}$, in which all the terms Δz_i are eliminated.

For each bidder i, the distribution of the highest bid of all his opponents is given by $\prod_{\lambda \neq i} \Phi^{(\lambda)}(x)$. In equation 11, we replace term Φ^{n-k}, with this new computation in order to find out the expect profit of each bidder i and then use this to compute the bid as given by equation 10. It should be noted that the bidding strategies of the bidders are no longer symmetric because of the difference in the knowledge about opponents' valuations.

Having described how to modify theorem 3 in order to compute the equilibria accurately, we now proceed to give the extension of theorem 2, for the case when the winners' bids are announced. In this case we know that all the remaining bids were lower than those prices.

Theorem 4. *Assume the same setting as theorem 3 with the difference that the auctions are now first price auctions and also that $\overrightarrow{p} = \{p_k\} (k = 1, \dots, j-1)$ are the discounted valuations that produced the closing prices observed.*[4] *In this scenario, it is a symmetric Bayes-Nash equilibrium strategy for each agent to bid:*

$$b_j^i = g_{j,j}\Big(\max\big\{0, v_j^i - \sum_{k=j+1}^{m} EP_{j,k}^i\big\}\Big) \tag{14}$$

where $\Phi_{j,k}()$, the pdf of any opponents' discounted valuation in the k^{th} auction, as computed during the j^{th} auction, is $\Phi_{j,k}() = \frac{N(x, \overrightarrow{p})}{D(\overrightarrow{p})}$, where these terms are computed by equations 12 and 13, and

$$EP_{j,k}^i = \begin{cases} \big(v_k^i - b_k^i - \sum_{\lambda=k+1}^{m} EP_{j,\lambda}^i\big)\Phi_{j,k}^{n-j}(g_{j,k}^{-1}(b_j^i)) & b_j^i > 0 \\ \frac{1}{n-k+1}\big(v_k^i - \sum_{\lambda=k+1}^{m} EP_{j,\lambda}^i\big)\Phi_{j,k}^{n-j}(0) & b_j^i = 0 \end{cases} \tag{15}$$

and the bidding strategy $g_{j,k}()$ that maps the discounted valuation to the bid in the k^{th} auction (as computed before the j^{th} auction takes place) is given by equation:

$$g_{j,k}(x) = x - \frac{1}{\Phi_{j,k}^{n-j}(x)} \int_{0^+}^{x} \Phi_{j,k}^{n-j}(\omega) d\omega \tag{16}$$

Proof. The proof is done by induction similar to the proof of theorem 3. The outer induction is on j, the number of the auction before which this computation is performed. When $j = 1$, no price information is known, so we have the case described in theorem 2 and it is easy to verify that the equations of this theorem give the equations of theorem 2.

Assuming that the theorem is correct when the computations are performed before auctions $1, \dots, j-1$, we need to show that the equations hold also when we perform

[4] As the bidding strategies $g_{\lambda,\lambda}()$ are known we can map any closing price p'_λ to the discounted valuation p_λ that produced this bid.

the same computations before the j^{th} auction is run. This inductive step is proved by a second induction, which is fairly similar to the proof of theorem 2. We need the equations to hold for any auction $k = j, \ldots, m$ considered. In the last (m^{th}) auction, there are $(N - m + 1)$ participating bidders, and we know that the symmetric Bayes-Nash equilibrium strategy in this case is to bid $b^i_m = v^i_m - \frac{1}{F_m^{n-m}(v^i_m)} \int_0^{v^i_m} F_m^{n-m}(\omega) d\omega$. [5] From this fact it follows that the equations hold when $k = m$. The inductive step is proven in more or less the same way as in the proof of theorem 2. The only difference is in the way that the distribution $\Phi_{j,k}(x)$ of any opponent's discounted valuation in the k^{th} auction as computed before the j^{th} auction, is generated. The latter is done in exactly the same way as in theorem 3. Note, in fact, that if the observed discounted valuations are the same in both auction variants, then the distributions $\Phi_{j,k}(x)$ would be identical. The rest of the proof follows the proof of theorem 2.

As far as revenue equivalence is concerned, it is easy to see that, if the observed discounted valuations are the same in both auction variants, then the distributions $\Phi_{j,k}(x)$ would be identical and then both the sequential first price and second price auctions would give the same revenue. However, in general this does not hold because of the different information provided by the closing prices of the two different auction variants:

Claim. Revenue equivalence does not hold between sequential first-price auctions and sequential second-price auctions.

5 Computing the Equilibria: An Example

In this section, we give an example case in order to further clarify the algorithm with which the bidding strategies are computed both when prices are announced and when they are not. We further show that, when the bidder who submitted the second price in each auction is known, we can use the fact that we know his discounted valuation precisely, in our analysis, and thus not make the assumption discussed at the beginning of section 4.

To keep it as simple as possible, we assume that there are $n = 3$ bidders and $m = 3$ items for sale in sequential second price auctions. The item valuations are independent and drawn from distributions F_1, which is uniform on $[0, 2]$, F_2, which is uniform on $[1, 2]$, and F_3, which is uniform on $[0, 1]$.

The first step both when prices are announced and when they are not is to compute the bidding strategies for all bidders before the first auction takes places. In this case the computation is the same in both cases. To be more precise, the computation when prices are not announced does not change as each auction is completed and thus we need only compute the bidding strategies once; on the other hand, this computation is valid before the first auction also when prices are announced, because at that point there is no price information yet.

Now, when the last auction would be reached, there would only be 1 bidder left, this bidder would bid

$$b_3 = v_3$$

and his expected profit in that auction will be

$$EP_3 = v_3$$

Given this information, in the second auction, where there would be 2 bidders left, they would bid $b_2 = v_2 - EP_3 \Leftrightarrow$

$$b_2 = v_2 - v_3$$

because it is always $v_2 \geq v_3$.

The distribution of the opponent bid in this auction has cdf which is computed by the following equation:

$$\Phi_2(x) = \int_0^1 F_2(x+\omega_3)F_3'(\omega_3)d\omega_3 = \int_0^1 F_2(x+\omega_3)d\omega_3$$

For $x > 2$, $\Phi_2(x) = 1$ and for $x < 0$, $\Phi_2(x) = 0$. We consider two cases:
If $x \in [0,1]$, then $F_2(x+\omega_3) = 0$, when $\omega_3 < 1 - x$, and $F_2(x+\omega_3) = x + \omega_3 - 1$, otherwise, thus

$$\Phi_2(x) = \int_{1-x}^1 (x+\omega_3-1)d\omega_3 = \frac{1}{2}x^2$$

If $x \in [1,2]$, then $F_2(x+\omega_3) = 1$, when $\omega_3 > 2 - x$, and $F_2(x+\omega_3) = x + \omega_3 - 1$, otherwise, thus

$$\Phi_2(x) = \int_0^{2-x} (x+\omega_3-1)d\omega_3 + \int_{2-x}^1 v d\omega_3 = -\frac{1}{2}x^2 + 2x - 1$$

Using this distribution, the expected profit of a bidder in that auction can be computed as:

$$EP_2 = \int_0^{v_2-v_3} (v_2 - \omega - v_3)\Phi_2'(\omega)d\omega$$

If $v_2 - v_3 \leq 1$, then this equation becomes:

$$EP_2 = \int_0^{v_2-v_3} (v_2 - \omega - v_3)\omega d\omega = \frac{1}{6}(v_2 - v_3)^3$$

If $v_2 - v_3 > 1$, then this equation becomes:

$$EP_2 = \int_0^1 (v_2 - \omega - v_3)\omega d\omega + \int_1^{v_2-v_3} (v_2 - \omega - v_3)(2-\omega)d\omega \Leftrightarrow$$

$$EP_2 = \frac{1}{6}(v_2 - v_3)^3 - \frac{1}{3}(v_2 - v_3 - 1)^3$$

Having computed the expected profit from the latter auctions, it is now possible to compute the bidders' bidding strategy in the first auction.

Any one of the 3 bidders would bid $b_1 = \max\{0, v_1 - EP_3 - EP_2\}$
If $v_2 - v_3 \leq 1$ then

$$b_1 = \max\{0, v_1 - v_3 - \frac{1}{6}(v_2 - v_3)^3\}$$

If $v_2 - v_3 > 1$ then

$$b_1 = \max\{0, v_1 - v_3 - \frac{1}{6}(v_2 - v_3)^3 + \frac{1}{3}(v_2 - v_3 - 1)^3\}$$

Let us now examine how the analysis would be changed if there are price announcements. First of all, the bid in the third auction would not change as it is a weakly dominant strategy to bid $b_3 = v_3$. Furthermore, $EP_{2,3} = v_3$ once more because in the last auction there is only one bidder remaining.[5] However, as a result of the price p being announced as the closing price of the first auction, we know that the bids of the bidders were $b_1 \leq p$. There are two cases:
If $p = 0$, then $b_1 = 0$ and thus we know that $v_1 - EP_3 - EP_2 \leq 0$ as the bid b_1 is the maximum of 0 and $v_1 - EP_3 - EP_2$.
If $p > 0$, then it is $v_1 - EP_3 - EP_2 \leq p$.
Thus in both cases it is $v_1 - EP_3 - EP_2 \leq p$. Note that according to our notation, $EP_2 = EP_{1,2}$ and $EP_3 = EP_{1,3}$, meaning that both these were computed before the 1^{st} auction; they have been computed already as described previously in this section.

We will now demonstrate how to compute the probability $\Phi_{2,2}(x)$ of the opponent bid in the second auction given that the closing price p in the first auction has been announced. We know that $\Phi_{2,2}(x) = \frac{N(x,p)}{D(p)}$, where $D(p)$ is the probability of the combinations of valuations that agree with the observed price p, and $N(x,p)$ is the probability of the combinations of valuations that not only agree with the observed price p, but also give a bid $b_2 \leq x$, where $b_2 = v_2 - v_3$ in this case since $EP_{2,3} = v_3$.

Notice that the probability of $v_1 \leq v_3 + EP_2 + p$ is $\frac{v_3+EP_2+p}{2}$. Thus the denominator $D(p)$ can be computed as equal to:[6]

$$D(p) = \int_1^2 \int_0^1 \frac{\omega_3 + EP_2 + p}{2} d\omega_3 d\omega_2 =$$

$$\int_1^2 \int_0^1 \frac{\omega_3 + \frac{(\omega_2-\omega_3)^3}{6} + p}{2} d\omega_3 d\omega_2 - \int_1^2 \int_0^{\omega_2 - 1} \frac{(\omega_2 - \omega_3 - 1)^3}{3} d\omega_3 d\omega_2$$

$$\Leftrightarrow D(p) = \frac{p}{2} + \frac{11}{30}$$

[5] Note that if that were not the case, then $EP_{2,3}$ would be different from $EP_{1,3}$, because the distribution $\Phi_{2,3}$ of any opponent's bid would also change.

[6] In this instance we chose to accelerate the computation by selecting one variable v_1 and removing it from the integration by selecting directly the probability of it satisfying the inequality, rather than using the indicator function $\mathcal{I}$ as in equation 13. This can be done in general for both $N()$ and $D()$ and for more than one inequalities. From each inequality $\omega_\lambda - \sum_{k=\lambda+1}^{m} EP^i_{\lambda,k} \leq p_\lambda$, we use the Bisection Method to find the value of v^i_k for which both sides of the inequality become equal (we take the algorithm from chapter 9 of [6]). This value v^{i*}_k is the smallest for which the inequality holds. It should also be noted that we are forced to discretize the computation even though the distribution $F_j()$ are continuous, when we use numerical methods to find these equilibria.

The nominator $N(x,p)$ can be computed, in a similar way. We distinguish two cases:
If $x \leq 1$, then we need to integrate for $v_2 \in [1, 1+x]$ and $v_3 \in [v_2 - x, 1]$ so that $v_2 - v_3 \leq x$ and thus:

$$N(x,p) = \int_1^{1+x} \int_{\omega_2 - x}^{1} \frac{\omega_3 + EP_2 + p}{2} d\omega_3 d\omega_2 =$$

$$\int_1^{1+x} \int_{\omega_2 - x}^{1} \frac{\omega_3 + \frac{(\omega_2-\omega_3)^3}{6} + p}{2} d\omega_3 d\omega_2 \Leftrightarrow$$

$$N(x,p) = \frac{1}{4}x^2 - \frac{1}{12}x^3 + \frac{1}{60}x^5 + \frac{1}{4}px^2$$

Therefore:

$$\Phi_{2,2}(x) = \frac{\frac{1}{4}x^2 - \frac{1}{12}x^3 + \frac{1}{60}x^5 + \frac{1}{4}px^2}{\frac{p}{2} + \frac{11}{30}}, x \leq 1$$

If $x > 1$, then we again distinguish several cases and get finally the following equation:

$$N(x,p) = \int_1^{2} \int_{\omega_2 - 1}^{1} \frac{\omega_3 + \frac{(\omega_2-\omega_3)^3}{6} + p}{2} d\omega_3 d\omega_2 +$$

$$\int_1^{x} \int_{0}^{\omega_2 - 1} \frac{\omega_3 + \frac{(\omega_2-\omega_3)^3}{6} - \frac{(\omega_2-\omega_3-1)^3}{3} + p}{2} d\omega_3 d\omega_2 +$$

$$\int_x^{2} \int_{\omega_2 - x}^{\omega_2 - 1} \frac{\omega_3 + \frac{(\omega_2-\omega_3)^3}{6} - \frac{(\omega_2-\omega_3-1)^3}{3} + p}{2} d\omega_3 d\omega_2 \Leftrightarrow$$

$$N(x,p) = -\frac{1}{2}p + \frac{4}{3}x - \frac{1}{2} - \frac{13}{12}x^2 - \frac{1}{6}x^4 + \frac{7}{12}x^3 + \frac{1}{60}x^5 - \frac{1}{4}px^2 + px$$

Therefore:

$$\Phi_{2,2}(x) = \frac{-\frac{1}{2}p + \frac{4}{3}x - \frac{1}{2} - \frac{13}{12}x^2 - \frac{1}{6}x^4 + \frac{7}{12}x^3 + \frac{1}{60}x^5 - \frac{1}{4}px^2 + px}{\frac{p}{2} + \frac{11}{30}}$$

when $x > 1$.

The final part of this example is to show how we can compute accurately $\Phi_{2,2}(x)$ when we know the identity of the bidder who placed a bid equal to p in the first auction.[7] In this case if $p = 0$ the analysis is still the same as before because it is $v_1 - EP_3 - EP_2 \leq 0$. However, when $p > 0$, then this means that for the opponent bidder who placed this bid it is: $v_1 - EP_3 - EP_2 = p$. To compute $\Phi_{2,2}(x)$ in this case we need to notice that both $D(p)$ and $N(x,p)$ are events that happen with probability 0 in our

[7] Actually since there are only two bidders left in the second auction, the one who did not place a bid equal to p in the first auction knows that the other did, even if it is not explicitly announced. But in general this information needs to be announced for the other bidders to be able to do this computation. Then, we don't make the assumption that $b \leq p$ for the bidder who set the closing price, and use the more accurate information that in fact his bid is $b = p$.

setting. To work around this problem we change $v_1 - EP_3 - EP_2 = p$ to $p \leq v_1 - EP_3 - EP_2 \leq p + \Delta p$ and take the limit as $\Delta p \to 0$. Then using the equations we have already computed, we can show that:

$$D(p) = \frac{1}{2}\Delta p$$

and if $x \leq 1$:

$$N(x,p) = \frac{1}{4}x^2 \Delta p$$

whereas if $x > 1$:

$$N(x,p) = -\frac{1}{2}\Delta p - \frac{1}{4}x^2 \Delta p + x\Delta p$$

Therefore, if $x \leq 1$:

$$\Phi_{2,2}(x) = \frac{1}{2}x^2$$

and, if $x > 1$:

$$\Phi_{2,2}(x) = -\frac{1}{2}x^2 + 2x - 1$$

Notice that this can be generalized to any number of prices $p_1, \ldots, p_{j-1}$, as was described in the previous section. More specifically, if it is known that a bidder has placed some bids equal to those prices, then we can compute D and N as before for the case of inequalities and then use these equations to compute the conditional probability Φ both in the case of equalities and inequalities in the same way; i.e. for each equality $b_\lambda = p_\lambda$, we change it to $p_\lambda \leq b_\lambda \leq p_\lambda + \Delta p_\lambda$ and take the limit as $\Delta p_\lambda \to 0$.

6 Discussion and Conclusions

In this paper we examined a sequence of first price or second price auctions each selling a single item. These items are partially substitutable in the sense that each bidder would bid on any one of them but only want to purchase one item in total. We initially gave the equilibrium strategies when the closing prices of auctions are not announced; under this setting we prove that using sequential first price auctions or sequential second price auctions yields the same expected revenue to the auctioneer. After that, we extended our analysis to include also the information obtained from announcements about the closing prices of the auctions; we can compute the equilibrium strategies in the first price variant accurately in all cases, but for the second price variant we need to also learn the identity of the bidder who submitted the second highest bid or disregard the fact that we know the actual bid of some bidder (if we don't known who this bidder was). Because the analysis is quite complicated, we then proceeded to give a small example to illustrate and clarify how the computation of the equilibria is executed both when prices are not announced and when they are.

Our next steps in this ongoing work is to find the optimal agenda, i.e. order of auctions that would maximize the revenue of the seller. Another avenue of research that we are currently pursuing is what happens if *free disposal* is possible in this scenario. It is somewhat restrictive to assume that once a buyer has purchased an item it cannot try to purchase any other for which it has a higher valuation. For example, in the case of a person buying paintings, even if she can only display one painting, she could still buy two and store the one not put on display. In this case, the fact that an item is purchased at some auction would decrease the bids for future items, because now the utility from obtaining a more valuable item is discounted by the valuation of the item that has been previously obtained. The full equilibrium analysis of this case is ongoing and will be presented in future work; however we give some of our initial results on computing the equilibrium in the appendix that follows. Another issue that we plan to examine is to determine the auction type that would yield the highest revenue for the seller in the case that prices are announced, as revenue equivalence does not hold in this case.

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A Free Disposal

In this section we give some initial results about what happens when free disposal is allowed. This means that, unlike in the previous work presented in this paper, when a bidder wins an item he is not forced to stop participating in the remaining auctions. The results that we present here are preliminary, because it is assumed at each auction round j, in the reasoning that each agent executes, that each one of the previous auctions has been won by a different bidder. Now, this is the most likely case, however, it is always possible that a single bidder might win two or more of the previous auctions, even though this event is fairly unlikely, given that the effective valuations of a bidder for the items sold in the remaining rounds are discounted by the valuation of the item that was last won, and therefore a bidder would have to have a pair of valuations v_α, v_β ($\alpha < \beta$), such that v_α is high enough to win the α^{th} auction and yet $v_\beta - v_\alpha$ is also win enough to win the β^{th} auction, considering that there will be bidders in that round who have not won any of the previous auctions and therefore their valuations are not discounted.

Having made this assumption, theorem 1 can be extended to allow for free disposal. We will use the notation $v_0 = 0$, i.e. we assume a dummy auction (with order number 0), before the first auction, for which everyone has valuation $v_0 = 0$. This will allow us to give one set of equation both for bidders who have won some previous auction and for those that have not; in the latter case we assume that the last auction that these bidders have won is the dummy auction 0. Additionally, $\overrightarrow{v^i} = v_1^i, \ldots, v_m^i$. Now, in this scenario, it is a symmetric Bayes-Nash equilibrium strategy for each agent to bid:[8]

$b_j^i = \max\left\{0, v_j^i - v_k^i + EP_{j+1}^i(\overrightarrow{v^i}, j) - EP_{j+1}^i(\overrightarrow{v^i}, k)\right\}$
when $v_j^i - v_k^i > 0$ and is otherwise:
$b_j^i = 0$,
where $EP_j^i(\overrightarrow{v^i}, k)$, the expected profit of bidder i from the j^{th} auction, when the most valued item that it has won was acquired by winning the k^{th} auction, is:
$EP_j^i(\overrightarrow{v^i}, k) = EP_{j+1}^i(\overrightarrow{v^i}, k) + \int_{0^-}^{b_j^i} \left(b_j^i - \omega\right) \frac{d}{d\omega}\left(\Phi_{j,0}^{n-j}(\omega) \prod_{\lambda=0,\lambda\neq k}^{j-1} \Phi_{j,\lambda}(\omega)\right) d\omega$
when $b_j^i > 0$. When $b_j^i = 0$ it is:
$EP_j^i(\overrightarrow{v^i}, k) = EP_{j+1}^i(\overrightarrow{v^i}, k) + \frac{\Phi_{j,0}^{n-j}(0) \prod_{\lambda=0,\lambda\neq k}^{j-1} \Phi_{j,\lambda}(0)}{n-j+1}\left(v_j^i - v_k^i + EP_{j+1}^i(\overrightarrow{v^i}, j) - EP_{j+1}^i(\overrightarrow{v^i}, k)\right)$
if $v_j^i > v_k^i$, and otherwise it is:
$EP_j^i(\overrightarrow{v^i}, k) = EP_{j+1}^i(\overrightarrow{v^i}, k)$
$\Phi_{j,k}()$, the pdf of any opponent bid in the j^{th} auction, when the most valued item was won in the k^{th} auction, is given by:

$$\Phi_{j,k}(x) = \underbrace{\int_0^\infty \ldots \int_0^\infty}_{(m-j+1)\ integrals} F_j(x + v_k^i - EP_{j+1}^i(\overrightarrow{v^i}, j) + EP_{j+1}^i(\overrightarrow{v^i}, k)) \cdot F_k^{'}(\omega_k) \underbrace{F_{j+1}^{'}(\omega_{j+1}) \ldots F_m^{'}(\omega_m)}_{(m-j)\ terms} \cdot d\omega_k \underbrace{d\omega_{j+1} \ldots d\omega_m}_{(m-j)\ vars}$$

The assumption that we made, which was that, at each auction round j, in the reasoning that each agent executes, each one of the previous auctions has been won by a different bidder, allows us to simplify the computation of the expected profit, in that we can then take the pdf of the distribution of the highest opponent bid to be equal to: $\Phi_{j,0}^{n-j}(\omega) \prod_{\lambda=0,\lambda\neq k}^{j-1} \Phi_{j,\lambda}(\omega)$. We believe that we should be able to remove this assumption, however this will lead to having to account for all possible combinations for which agents could have won each combination of auctions in the past and this will complicate significantly the analysis.

Using similar reasoning, we can show that *the first-price auction variant of this scenario is revenue equivalent to the second-price auction setting* that was just pre-

[8] We don't give the proof here, but some of the main ideas are similar to the proof of theorem 1, although there are some additional steps given the extra complexity that the valuations of a bidder are reduced in the remaining auctions, if it wins any auction, whereas in the scenario of theorem 1, the valuations for the remaining rounds effectively become 0 (as the agent must drop out).

sented. Using this knowledge, we can extend these results to compute the Bayes-Nash equilibrium for the case of sequential first-price auctions with free disposal.