Considering Assymmetric Opponents In Multi-Unit Sealed-Bid Auctions

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In this paper, we examine two important multi-unit auction scenarios that have been looked at in the auction literature: the case of bidders having any risk attitude, and of competitive bidders who are not simply self-interested, but also wish to beat the other participants. In the existing work, a symmetric model of all bidders has been assumed; they all have the same risk attitude and the same competitiveness respectively. We extend the existing literature by relaxing this assumption and allowing each bidder to have a risk attitude (resp. competitiveness) that is potentially different from others, and which is known a priori only to himself. For these cases, we examine the Bayes-Nash equilibria in both m^{th} and $(m+1)^{th}$ price sealed-bid auctions. We then present a method and an algorithm for solving the systems of differential equations which characterize the Bayes-Nash equilibria and computing the equilibrium strategies.

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1. INTRODUCTION

Auctions have become commonplace and they are used to trade all kinds of commodity both between governments and companies, as well as private individuals. Game theory is widely used in such multi-agent scenarios, as a way to model and predict the behavior of bidders participating in these auctions. The scenarios normally analyzed in the auction literature assume in almost all cases that the bidders participating in the auction are symmetric in the sense that they have their parameters (i.e. their valuations for the goods they bid to buy) drawn from the same prior distributions and have the same utility model. There is very little work that relaxes this assumption. More specifically, [Maskin and Riley 2000] and [Lebrun 1999] compute equilibria for auctions with asymmetric bidders with different prior distributions from which their valuations are drawn, and an experimental evaluation is conducted in [Gutha et al. 2005].

In this paper, we examine two important multi-unit auction scenarios that have been looked at in the auction literature: the case of bidders with any risk attitude, and of competitive bidders. In the existing work, a symmetric model of all bidders has been assumed; they all have the same risk attitude thus using the same utility function, and they all have the same competitiveness respectively. For example, in [Liu et al. 2003] and [Krishna 2002], cases where agents are not risk neutral, but rather risk averse, are examined. In all instances, the agents are assumed to be risk averse in exactly the same way, and they all have the same utility function, which maps profit to utility in exactly the same way. In [Brandt et al. 2007] and [Vetsikas and Jennings 2007] a different kind of utility function is assumed; the bidders in this case not only wish to maximize their own profit, but they also wish to minimize the opponents' profit; these two goals are weighted by the agent's spite coefficient, which determines the relative importance assigned to these goals. In both papers, the model of the agents is the same, in the sense that they all use the same utility function, and the same spite coefficient.

To extend these results, we introduce asymmetries in the bidders' models. However, unlike in [Lebrun 1999], where the models of all bidders are common knowledge, here we assume that each bidder only knows his own model, i.e. his utility function (resp. spite coefficient), in the case of bidders not being risk neutral, and that the opponents use a variety of models each with a certain known probability. For example, a bidder knows how competitive he is, but not how competitive the opponents are; he does know however that there is a certain chance associated with each opponent using a particular model (i.e. competition coefficient in this case).

This paper is organized as follows. In the next section, we formally present the model and the notation that will be used in this paper. Then, in sections 3 and 4, we derive the systems of differential equations that characterize the Bayes-Nash equilibria that exist in the case of uncertainty in the opponents risk attitude and their competitiveness respectively. As these systems of differential equations are quite complicated and they don't correspond precisely to any system with a known solution, in section 5 we show how to generate the equilibria; we provide a methodology (and algorithms) for solving systems of differential equations such as these. We give three examples. We start with a simpler example with competitiveness uncertainty in section 5.1 and then, we present our methodology in section 5.2, and use it to compute a more complicated case in section 5.3. Then, we show how the methodology is used to compute the equilibrium strategy when there is uncertainty in the bidders' risk attitudes in section 5.4. Finally, we conclude.

2. THE MULTI-UNIT AUCTION SETTING

In this section we formally describe the auction setting to be analyzed and define the objective function that the agents wish to maximize. We also give the notation that we use.

In particular, we will compute Bayes-Nash equilibria for sealed-bid auctions where $m \geq 1$ identical items are being sold; these equilibria will be defined by a set of strategies $g_\alpha(v)$, which map the agents' valuations v_i to bids b_i. These strategies are parameterized by a parameter α, which will indicate the model of the agent, i.e. his risk attitude, or his competitiveness factor (depending on the scenario that we examine). Thus we assume that two agents will use the same bidding strategy, if they have the same model (same parameter α). The two most common multi-unit auction settings are the m^{th} and $(m+1)^{th}$ price auctions, in which the top m bidders win one item each at a price equal to the m^{th} and $(m+1)^{th}$ highest bid respectively.

More specifically, we assume that N indistinguishable bidders (where $N \geq m$) participate in the auction and each has a private valuation (utility) v_i for acquiring any one[1] of the traded items, which is known only to himself; these valuations are assumed to be i.i.d. drawn from a distribution with cumulative distribution function (cdf) $F(v)$, which is the same for all bidders. Furthermore, we assume that $F(v)$ has support in $[v_l, v_h]$, which means that $\forall v < v_l \vee v > v_h : F'(v) = 0$.[2] Let $\widetilde{u}_i$ be the profit of agent i (i.e. $\widetilde{u}_i = 0$, if it does not win an item, and $\widetilde{u}_i = v_i - p_i$, if it does) and p_i is the total payment the agent must make to the auctioneer. Each agent has uncertainty not only for the opponents' valuations v_j, but also for their model (i.e. their parameters α_j). They know however the prior distribution $H(\alpha)$ from which each opponent parameter α is drawn. So even though each agent knows only it's own model, it can make a probabilistic inference on the possible opponent models.

In the first scenario (analyzed in section 3), we examine self-interested agents with varying risk attitudes. The possible risk attitudes belong to a family of utility functions $u_\alpha()$, which are characterized by the parameter α. Thus, we assume that the objective function (i.e. the total utility) that each agent tries to maximize depends only on his own gain $\widetilde{u}_i$, and is equal to:

$$U_i = u_{\alpha_i}(\widetilde{u}_i)$$

Some families of utility functions $u_\alpha(x)$ used widely in economics are: $u_\alpha(x) = x^\alpha, \alpha \in (0,1)$ (CRRA), and $u_\alpha(x) = 1 - exp(-\alpha x), \alpha > 0$ (CARA), both of which indicate risk-averse bidders.

In the second scenario, the agents are now risk-neutral, but they are competitive, rather than self-interested. This means that they not only care about maximizing their profit, but also about minimizing the profit of the opponents. We define the objective function of each agent in the

[1] We make the assumption here that each bidder is interested in exactly one item; this is a usual assumption made in the analysis of multi-unit auctions, as the analysis even for self-interested risk-neutral bidders which are interested in purchasing multiple copies of the item is an open problem.

[2] For example, if $F(u)$ is the uniform distribution $U[0,1]$, then $v_l = 0$ and $v_h = 1$. If $F(v)$ is such that v_i can take values in $[0, +\infty)$, then $v_l = 0$ and $v_h = +\infty$. We use these tight bounds to define the boundary conditions of some of the equilibria we will compute.

same way as in [Brandt et al. 2007; Vetsikas and Jennings 2007]:

$$U_i = (1-\alpha_i)\widetilde{u}_i - \alpha_i \sum_{j\neq i} \widetilde{u}_j$$

where $\alpha_i \in [0,1]$ is a parameter called the *competition (or spite) coefficient*, which denotes the degree of competition of agent i; the higher it is the more the agent cares about minimizing the opponent profit, rather than maximizing his own.

We also use the following additional notation in the proofs:

$$\Phi_k(x) = \sum_{i=0}^{k-1} C(N-1,i)x^{N-1-i}(1-x)^i \quad (1)$$

where the notation $C(n,k)$ is the total number of possible combinations of k items chosen from n. This formula is useful because, if $Z(x)$ is the probability distribution of any opponent's bid b_j, i.e. $Z(x) = Prob[b_j \le x]$, and $B^{(k)}$ is the k^{th} order statistic of these bids of the opponents, then the distribution of $B^{(k)}$ is: [Rice 1995]

$$Prob[B^{(k)} \le x] = \Phi_k(Z(x)) \quad (2)$$

As shown in [Vetsikas and Jennings 2007], for all N and m, such that $N \ge m$ the following equations hold:

$$\Phi'_m(x) = (N-m)\big(\Phi_m(x) - \Phi_{m-1}(x)\big)\frac{1}{x} \quad (3)$$

$$\Phi'_m(x) = m\big(\Phi_{m+1}(x) - \Phi_m(x)\big)\frac{1}{1-x} \quad (4)$$

We will use equations 1, 2 and 3 in the computation of the equilibria. To reduce the size of some equations in the proofs, let us also define:

$$\Delta\Phi_m(x) = \Phi_m(x) - \Phi_{m-1}(x) \quad (5)$$

3. EQUILIBRIA WHEN OPPONENT RISK ATTITUDES ARE NOT KNOWN

In this section, we assume that each agent has uncertainty not only for the opponents' valuations, but also for their risk attitudes. The possible risk attitudes belong to a family of functions, which are characterized by an one dimensional parameter α, which is drawn from a known probability distribution (H). We therefore assume that each agent i knows its own valuation v_i and risk attitude function $u_{\alpha_i}()$, and also the distributions $F(v)$ and $H(\alpha)$ from which the valuations v and risk attitude functions $u_\alpha()$ of the other agents are drawn.

THEOREM 1. *In the case of an m^{th} price sealed-bid auction with N participating bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a risk attitude described by utility function $u_{\alpha_i}()$, where v_i and α_i are i.i.d. random variables drawn from distributions $F(v)$ and $H(\alpha)$ respectively, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $g_{\alpha_i}(v_i)$ is the solution of the system of differential equations:*

$$\forall v_i, \alpha_i : (N-m)\int_{-\infty}^{+\infty} \frac{F'(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))} H'(\alpha)d\alpha = \quad (6)$$

$$\frac{u'_{\alpha_i}(v_i - g_{\alpha_i}(v_i))}{u_{\alpha_i}(v_i - g_{\alpha_i}(v_i)) - u_{\alpha_i}(0)} \int_{-\infty}^{+\infty} F(g_\alpha^{-1}(g_{\alpha_i}(v_i)))H'(\alpha)d\alpha$$

with boundary conditions: $g_{\alpha_i}(v_l) = v_l, \forall \alpha_i$.

PROOF. We assume that the equilibrium strategy is described by functions $g_\alpha(v)$ which map the valuations v to bids for any of the possible risk attitude functions $u_\alpha()$. We use this knowledge to determine the bids of the opponents and the expected profit that a bidder i gets from placing a bid equal to b_i. The distribution from which an opponent's bid b_j is drawn has cdf: $Prob[b_j \le$

$x|\alpha_j] = F(g^{-1}_{\alpha_j}(x))$, when his risk attitude is described by function $u_{\alpha_j}()$. Therefore, using Bayes' rule we compute this probability for any possible value of α_j:

$$Prob[b_j \leq x] = \int_{-\infty}^{+\infty} F(g_\alpha^{-1}(x))H'(\alpha)d\alpha \tag{7}$$

The distribution of the k^{th} highest opponent bid $B^{(k)}$, as there are $(N-1)$ opponents, is :

$$Prob[B^{(k)} \leq x] = \Phi_k\Big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(x))H'(\alpha)d\alpha\Big) \tag{8}$$

where $\Phi_k(x)$ is given by equation 1.

We can now analyze the expected profit of bidder i. Let b_i be the bid that he places in the auction. We distinguish the following cases:

(i) If $b_i < B^{(m)}$, then bidder i is outbid and doesn't win any items, therefore his utility is $u_i = u_{\alpha_i}(0)$.

(ii) If $B^{(m)} \leq b_i \leq B^{(m-1)}$, then bidder i has placed the last winning bid. Thus the payment equals his bid and his utility is $u_i = u_{\alpha_i}(v_i - b_i)$. The probability of this case happening is: $Prob[B^{(m)} \leq b_i \leq B^{(m-1)}] = \Delta\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)$.

(iii) If $B^{(m-1)} < b_i$, then bidder i is a winner, the payment is equal to bid $B^{(m-1)}$ and his utility is $u_i = u_{\alpha_i}(v_i - B^{(m-1)})$. Note that: $Prob[B^{(m-1)} \leq \omega] = \Phi_{m-1}\Big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(\omega))H'(\alpha)d\alpha\Big)$.

The expected utility of bidder i, who places bid b_i, is:

$$\begin{aligned} Eu_i(b_i) =& u_{\alpha_i}(0)\Big(1 - \Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)\Big) \\ &+ u_{\alpha_i}(v_i - b_i)\Delta\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big) \\ &+ \int_0^{b_i} u_{\alpha_i}(v_i - \omega)\frac{d}{d\omega}\Big(\Phi_{m-1}\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(\omega))H'(\alpha)d\alpha\big)\Big)d\omega \end{aligned} \tag{9}$$

The bid which maximizes this expected utility, is found by setting: $\frac{dEu_i}{db_i} = 0$. This becomes:

$$(u_{\alpha_i}(v_i - b_i) - u_{\alpha_i}(0))\frac{d}{db_i}\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)$$

$$= u'_{\alpha_i}(v_i - b_i)\Delta\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)$$

Using equation 3, to simplify this equation, we derive:

$$(N-m)\frac{\frac{d}{db_i}\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)}{\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha} = \frac{u'_{\alpha_i}(v_i - b_i)}{u_{\alpha_i}(v_i - b_i) - u_{\alpha_i}(0)}$$

This value b_i is equal to $b_i = g_{\alpha_i}(v_i)$, since it maximizes the expected utility $Eu_i(b_i)$. Using this substitution, we derive the system of differential equations 6 for all possible values of v_i, α_i. The boundary conditions come from the fact that a bidder with valuation $v_i = v_l$ (i.e. the lowest possible valuation) will always bid $b_i = v_l$.

COROLLARY 1. *In the case that $H(\alpha)$ is (or can be approximated by) a discrete distribution, which has support in X_H,[3] the system of differential equations 6 becomes:*

$$\forall v_i, \alpha_i: \quad (N-m)\sum_{\alpha \in X_H} \frac{F'(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}h(\alpha) = \tag{10}$$

[3] Thus $h : X_H \to [0,1]$, where $X_H = \{\alpha | h(\alpha) > 0\}$.

$$\frac{u'_{\alpha_i}(v_i - g_{\alpha_i}(v_i))}{u_{\alpha_i}(v_i - g_{\alpha_i}(v_i)) - u_{\alpha_i}(0)} \sum_{\alpha \in X_H} F(g_\alpha^{-1}(g_{\alpha_i}(v_i)))h(\alpha)$$

where $h(x) = Prob[\alpha = x]$ is the (discrete) probability function of the distribution H.

For the case of an $(m+1)^{th}$ price auction, it is still a (weakly) dominant strategy to bid truthfully, i.e. use the same strategy as in the case when all the bidders use the same model:

FACT 1. *In the case of an $(m+1)^{th}$ price sealed-bid auction with N participating bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a risk attitude described by utility function $u_{\alpha_i}()$, it is a (weakly) dominant strategy to bid truthfully : $b_i = v_i$.*

4. EQUILIBRIA WHEN OPPONENT COMPETITIVENESS ARE NOT KNOWN

In this section, we assume that each agent has uncertainty not only for the opponents' valuations, but also for how competitive they are. The competitiveness of an agent is characterized by his competition coefficient α_i, which takes values in $[0, 1]$, which is drawn from a known probability distribution $H(\alpha)$. We therefore assume that each agent i knows its own valuation v_i and competition coefficient α_i, and also the distributions F and H from which the valuations and competition coefficients of the other agents are drawn.

THEOREM 2. *In the case of an m^{th} price sealed-bid auction with N participating risk-neutral bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a competition coefficient α_i, where v_i and α_i are i.i.d. random variables drawn from distributions $F(v)$ and $H(\alpha)$ respectively, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $g_{\alpha_i}(v_i)$ is the solution of the system of differential equations:*

$$\forall v_i, \alpha_i : \frac{1 - \alpha_i m}{N - m} \int_{-\infty}^{+\infty} F(g_\alpha^{-1}(g_{\alpha_i}(v_i)))H'(\alpha)d\alpha = \tag{11}$$

$$\int_{-\infty}^{+\infty} \frac{F'(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))} \big(v_i - g_{\alpha_i}(v_i) - \alpha_i v_i + \alpha_i g_\alpha^{-1}(g_{\alpha_i}(v_i))\big) H'(\alpha)d\alpha$$

with boundary conditions: $g_{\alpha_i}(v_l) = v_l, \forall \alpha_i$.

PROOF. We assume that the equilibrium strategy is described by functions $g_\alpha(v)$ which map the valuations v to bids for any of the competition factors α. We will use this knowledge to determine the bids of the opponents and the expected profit that a bidder i gets from placing a bid equal to b_i. The distributions of any one opponent bid b_j and of the k^{th} highest opponent bid $B^{(k)}$ are given from equations 7 and 8 (use the same reasoning as in theorem 1). Now, bidder i bids b_i, the bid that maximizes his objective function on expectation.

Let C be the sum (on expectation) of the $(m-1)$ opponent valuations that produced the top $m-1$ winning bids. Since in all cases that we will examine, whether bidder i wins or not, we know that the opponents with the top $(m-1)$ bids will each win an item, we know that they will gain this amount C from doing so. This value is a constant and does not depend on the bid b_i. We will mostly ignore this term in the rest of the computations.

Depending on the bid b_i, we need to consider the following three cases:

(i) When $B^{(m)} > b_i$, bidder i does not win any item and the closing price is $B^{(m)}$. Therefore bidder i's gain is 0 and the opponents make a gain from gaining an extra item (the m^{th}), in addition to the $(m-1)$ items that they always win (this was counted in the constant value C). We must compute the expected gain obtained by getting this extra item. Let us assume that the actual value of $B^{(m)} = x$. This is equal to a bid submitted by an agent (w.l.o.g. assume this is agent j). Then $b_j = x$ and we want to find out the expectation of the value of the valuation v_j that generated this bid for all possible values of α_j. Let us denote this by $EV(x) = E(v_j | b_j = x)$. For a particular value of α_j, i.e. when $\alpha_j = \alpha$, we know that $v_j = g_\alpha^{-1}(x)$ and this happens with probability $Prob[b_j = x|\alpha_j = \alpha] = \frac{d}{dx}F(g_\alpha^{-1}(x))$. Using Bayes rule we can compute the value of $E(v_j|b_j = x)$ being equal to:

$$EV(x) = \frac{\int_{-\infty}^{+\infty} g_\alpha^{-1}(x)\frac{d}{dx}F(g_\alpha^{-1}(x))H'(\alpha)d\alpha}{\int_{-\infty}^{+\infty}\frac{d}{dx}F(g_\alpha^{-1}(x))H'(\alpha)d\alpha} \tag{12}$$

They also must make total payments of $mB^{(m)}$. The total additional[4] expected utility for bidder i in this case is hence:

$$\Delta U_1 = \alpha_i \int_{b_i}^{\infty} \big(m\omega - EV(\omega)\big)\frac{d}{d\omega}\big(\Phi_m(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(\omega))H'(\alpha)d\alpha)\big)d\omega \tag{13}$$

(ii) When $B^{(m-1)} > b_i \geq B^{(m)}$, bidder i wins an item and the closing price is b_i. Therefore bidder i's gain is $v_i - b_i$ and the opponents pay $(m-1)b_i$ for the items that they win. The total additional expected utility for bidder i is:

$$\Delta U_2 = \big((1-\alpha_i)(v_i - b_i) + \alpha_i(m-1)b_i\big)\Delta\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big) \tag{14}$$

(iii) When $b_i \geq B^{(m-1)}$, bidder i wins an item and the closing price is $B^{(m-1)}$. Therefore bidder i's gain is $v_i - B^{(m-1)}$ and the opponents must pay $(m-1)B^{(m-1)}$ for the items that they purchase. The total additional expected utility for bidder i in this case is:

$$\Delta U_3 = \int_0^{b_i} \big((1-\alpha_i)(v_i - \omega) + \alpha_i(m-1)\omega\big)\frac{d}{d\omega}\big(\Phi_{m-1}(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(\omega))H'(\alpha)d\alpha)\big)d\omega \tag{15}$$

The total expected utility for bidder i when considering all possibilities is therefore: $EU_i(b_i) = -\alpha_i C + \Delta U_1 + \Delta U_2 + \Delta U_3$. To find the value of v_i that maximizes the expected utility $EU_i(b_i)$, we set $\frac{dEU_i(b_i)}{db_i} = 0$. We then get:

$$(1-\alpha_i m)\Delta\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big) = \tag{16}$$

$$\big(v_i - b_i - \alpha_i v_i + \alpha_i EV(b_i)\big)\frac{d}{db_i}\Big(\Phi_m\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)\Big)$$

By using equation 3 to simplify equation 16, we get:

$$\frac{1-\alpha_i m}{N-m}\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha = \tag{17}$$

$$\big(v_i - b_i - \alpha_i v_i + \alpha_i EV(b_i)\big)\int_{-\infty}^{+\infty}\frac{d}{db_i}F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha$$

Since strategy $g_\alpha(v)$ gives the equilibrium strategy, then it must be the case that the value of b_i that maximizes the total utility is given by $g_\alpha(v)$ (i.e. that $b_i = g_\alpha(v_i)$). Using this fact and equation 12 to substitute in equation 17, we get equation 11.

We select the boundary condition $g_\alpha(v_l) = v_l$, based on the fact that this boundary condition holds when all the agents have the same competition factor α, $\forall\alpha$.[5] (see [Vetsikas and Jennings 2007])

[4] We mean additional to the fact that the agent always loses utility αC, since its opponents always gain a value C from the top $(m-1)$ items.

[5] As it is necessary to know the value of $g_\alpha(v), \forall\alpha$ at the same point $\hat{v}$, we selected this boundary condition, but we still need to double check that it produces the correct equilibria.

COROLLARY 2. *In the case that $H(\alpha)$ is (or can be approximated by) a discrete distribution, which has support in X_H, the system of differential equations 11 becomes:*

$$\forall v_i, \alpha_i : \frac{1-\alpha_i m}{N-m} \sum_{\alpha \in X_H} F(g_\alpha^{-1}(g_{\alpha_i}(v_i)))h(\alpha) = \tag{18}$$

$$\sum_{\alpha \in X_H} \frac{F'(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))} \big(v_i - g_{\alpha_i}(v_i) - \alpha_i v_i + \alpha_i g_\alpha^{-1}(g_{\alpha_i}(v_i))\big) h(\alpha)$$

where $h(x) = Prob[\alpha = x]$ is the (discrete) probability function of the distribution H.

THEOREM 3. *In the case of an $(m+1)^{th}$ price sealed-bid auction with N participating risk-neutral bidders, in which each bidder i is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to v_i, and has a competition coefficient α_i, where v_i and α_i are i.i.d. random variables drawn from distributions $F(v)$ and $H(\alpha)$ respectively, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $g_{\alpha_i}(v_i)$ is the solution of the system of differential equations:*

$$\forall v_i, \alpha_i : -\alpha_i \Big(1 - \int_{-\infty}^{+\infty} F(g_\alpha^{-1}(g_{\alpha_i}(v_i)))H'(\alpha)d\alpha\Big) = \tag{19}$$

$$\int_{-\infty}^{+\infty} \frac{F'(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))} \big(v_i - g_{\alpha_i}(v_i) - \alpha_i v_i + \alpha_i g_\alpha^{-1}(g_{\alpha_i}(v_i))\big) H'(\alpha)d\alpha$$

with boundary conditions: $g_{\alpha_i}(v_h) = v_h, \forall \alpha_i$.

PROOF. Once more, we assume that the equilibrium strategy is described by functions $g_\alpha(v)$ which map the valuations v to bids for any of the competition factors α. We will use this knowledge to determine the bids of the opponents and the expected profit that a bidder i gets from placing a bid equal to b_i. The distributions of any one opponent bid b_j and of the k^{th} highest opponent bid $B^{(k)}$ are given from equations 7 and 8 (use the same reasoning as in theorem 1). Now, bidder i bids b_i, the bid that maximizes his objective function on expectation.

Let C be the sum (on expectation) of the m opponent valuations that produced the top m winning bids. In most of the cases that we will examine (when bidder i does not win), the opponents with the top m bids will each win an item, and they will gain this amount C from doing so. This value is a constant and does not depend on the bid b_i. In the case that bidder i outbids the competition (case iii below), then we will subtract the expected valuation that produced the m^{th} highest opponent bid from the opponents' gain in order to compensate.

Depending on the bid b_i, we need to consider the following three cases:

(i) When $B^{(m+1)} > b_i$, bidder i does not win any item and the closing price is $B^{(m+1)}$. Therefore bidder i's gain is 0 and the opponents must make total payments of $mB^{(m+1)}$. The total additional expected utility for bidder i in this case is hence:

$$\Delta U_1 = \alpha_i \int_{b_i}^{\infty} m\omega \frac{d}{d\omega} \big(\Phi_{m+1}(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(\omega))H'(\alpha)d\alpha)\big) d\omega \tag{20}$$

(ii) When $B^{(m)} > b_i \geq B^{(m+1)}$, bidder i does not win and the closing price is b_i. Therefore bidder i's gain is 0 and the opponents must pay mb_i for the items that they purchase. The total additional expected utility for bidder i in this case is:

$$\Delta U_2 = \alpha_i m b_i \Delta\Phi_{m+1}\big(\int_{-\infty}^{+\infty} F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big) \tag{21}$$

(iii) When $b_i \geq B^{(m)}$, bidder i wins an item and the closing price is $B^{(m)}$. Therefore bidder i's gain is $v_i - B^{(m)}$ and the opponents must pay $(m-1)B^{(m)}$ for the items that they purchase. We also need to subtract the valuation of the m^{th} highest bid $B^{(m)}$ from the total gain of the opponents, since they only won $(m-1)$ items. This expected utility $EV(x)$, when $B^{(m)} = x$, is given by equation 12. Thus the total additional utility in this case is:

$$\Delta U_3 = \int_0^{b_i} \big((1-\alpha_i)(v_i-\omega)+\alpha_i((m-1)\omega+EV(\omega))\big)\frac{d}{d\omega}\big(\Phi_m\big(\int_{-\infty}^{+\infty}F(g_\alpha^{-1}(\omega))H'(\alpha)d\alpha\big)\big)d\omega \tag{22}$$

The total expected utility for bidder i when considering all possibilities is therefore: $EU_i(b_i) = -\alpha_i C + \Delta U_1 + \Delta U_2 + \Delta U_3$. To find the value of v_i that maximizes the expected utility $EU_i(b_i)$, we set $\frac{dEU_i(b_i)}{db_i} = 0$. We then get:

$$-\alpha_i m \Delta\Phi_{m+1}\big(\int_{-\infty}^{+\infty}F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big) = \tag{23}$$

$$\big(v_i - b_i - \alpha_i v_i + \alpha_i EV(b_i)\big)\frac{d}{db_i}\Big(\Phi_m\big(\int_{-\infty}^{+\infty}F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\big)\Big)$$

By using equation 4 to simplify equation 23, we get:

$$-\alpha_i\Big(1-\int_{-\infty}^{+\infty}F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha\Big) = \tag{24}$$

$$\big(v_i - b_i - \alpha_i v_i + \alpha_i EV(b_i)\big)\int_{-\infty}^{+\infty}\frac{d}{db_i}F(g_\alpha^{-1}(b_i))H'(\alpha)d\alpha$$

Since strategy $g_\alpha(v)$ gives the equilibrium strategy, then it must be the case that the value of b_i that maximizes the total utility is given by $g_\alpha(v)$ (i.e. that $b_i = g_\alpha(v_i)$). Using this fact and equation 12 to substitute in equation 24, we get equation 19.

To select the appropriate boundary condition one should note that when everyone else bids $g(v_h) = v_h$, then no one bidder j with value $v_j = v_h$ can gain more utility by increasing his bid b_j above v_h. Also we know that when bidder j has $\alpha_j = 0$, he bids truthfully, and when $\alpha > 0$, he bids at least his true value. Thus a bidder with value $v_j = v_h$, will bid $b_j = v_h$; thus we get the boundary condition.

COROLLARY 3. *In the case that $H(\alpha)$ is (or can be approximated by) a discrete distribution, which has support in X_H, the system of differential equations 19 becomes:*

$$\forall v_i, \alpha_i : -\alpha_i\Big(1-\sum_{\alpha\in X_H}F(g_\alpha^{-1}(g_{\alpha_i}(v_i)))h(\alpha)\Big) = \tag{25}$$

$$\sum_{\alpha\in X_H}\frac{F'(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}\big(v_i - g_{\alpha_i}(v_i) - \alpha_i v_i + \alpha_i g_\alpha^{-1}(g_{\alpha_i}(v_i))\big)h(\alpha)$$

where $h(x) = Prob[\alpha = x]$ is the (discrete) probability function of the distribution H.

5. THE METHODOLOGY FOR COMPUTING THE BAYES-NASH EQUILIBRIA

In this section, we demonstrate how to solve the differential equations of the theorems presented in the previous sections. In particular, we show how to compute the equilibria using some representative cases. Initially, we give an example, in section 5.1, where the system of differential equations can be decomposed immediately and solved using standard differential equation solvers. Then, in section 5.2, we give a method and an algorithm based on that method for solving the general system of differential equations in cases where it cannot be decomposed. We then use this method in section 5.3, in order to solve a general example. Then, we the methodology to compute the equilibrium strategy of an example where there is uncertainty in the bidders' risk attitudes in section 5.4. Note that, while the algorithm and method themselves would work for any distribution in the way we describe, in the specific examples used in this section, we will choose $F(v)$ to be the uniform distribution $U[0,1]$.

5.1 Example 1: Asymmetric Competitiveness

We begin with a simple example. We analyze the case of an $(m+1)^{th}$ price auction. In this instance the number of bidders N and of the items sold m does not matter for the equilibrium strategy. We are going to assume that there are two possible types and thus a bidder can be either self-interested ($\alpha = 0$), with probability $(1-p)$, or be competitive to some specified degree (which is characterized by coefficient $\alpha > 0$), with probability p.

Since equation 25 must hold $\forall \alpha_i$, we set $\alpha_i = 0$ and $\alpha_i = \alpha$ in it. Setting $\alpha_i = 0$ eventually gives $g_0(v) = v$. This was expected as a self-interested bidder participating in an $(m+1)^{th}$ price auction has (weakly) dominant strategy to bid truthfully.[Krishna 2002] Therefore our analysis yields the expected result in this case. Now, we can use the fact that a self-interested bidder bids truthfully, to simplify the equation obtained from 25 when setting $\alpha_i = \alpha$. In the end the solution is given by the differential equation: (for $\forall \alpha, p \in [0,1]$)

$$g_{\alpha}^{'}(v) = p\Big(\alpha\frac{(1-p)g_{\alpha}(v)\ +\ p\,v\ -\ 1}{v - g_{\alpha}(v)} - (1-p)(1-\alpha)\Big)^{-1} \tag{26}$$

Thus:

$$g_{\alpha}(v) = \beta v + 1 - \beta \tag{27}$$

where:

$$\beta = \frac{1-\alpha-2\,p+\sqrt{1-2\,\alpha+\alpha^2+4\,\alpha\,p}}{2\,(1-p)} \tag{28}$$

This is the unique equilibrium for this setting, and it exists for $\forall \alpha, p \in [0,1]$. It is interesting to note that a bidder with coefficient α will bid more in this case, in which there is a chance to go up against bidders which are strictly self-interested (i.e. $\alpha = 0$), than when all bidders have the same α.[6] This can be explained by the following fact: because the opponents have a chance to bid truthfully and therefore less than the bidders with $\alpha > 0$, the closing price would be decreased and this would make opponents pay less and thus make more profit. Thus the competitive bidders would bid higher to counteract this effect.

5.2 An Methodology for Computing the Equilibria in the General Case

In the previous section the solution of the system of differential equations was made easy by the fact that there were only two possible bidder models and in one of them the bidders were bidding truthfully. In the general setting, this will not happen. By examining the system of differential equations that need to be solved in corollaries 1, 2 and 3, we note that all of them have the same general form:[7]

$$\frac{\Psi_{j,1}\Big(g_{\alpha_1}^{-1}(g_{\alpha_j}(v_i)),\ldots,g_{\alpha_\lambda}^{-1}(g_{\alpha_j}(v_i))\Big)}{g'_{\alpha_1}(g_{\alpha_1}^{-1}(g_{\alpha_j}(v_i)))} + \ldots + \frac{\Psi_{j,\lambda}\Big(g_{\alpha_1}^{-1}(g_{\alpha_j}(v_i)),\ldots,g_{\alpha_\lambda}^{-1}(g_{\alpha_j}(v_i))\Big)}{g'_{\alpha_\lambda}(g_{\alpha_\lambda}^{-1}(g_{\alpha_j}(v_i)))} =$$

$$\Psi_{j,0}\Big(g_{\alpha_j}(v_i), g_{\alpha_1}^{-1}(g_{\alpha_j}(v_i))\ldots,g_{\alpha_\lambda}^{-1}(g_{\alpha_j}(v_i))\Big), \forall j = 1,\ldots,\lambda \tag{29}$$

where $\Phi_{j,k}()$ can be any function with the arguments noted in the formulas. In order to solve this system, we need to separate the derivatives $g'_{\alpha_k}()$.[8] We can thus decompose this system into a system that has similarities with the standard systems of ordinary differential equations that an algorithm like the Euler (or the Runge-Kutta) method would solve.

The methodology we propose for solving any system like this has the following steps:

[6] In this latter case the bidding strategy is $g_\alpha(v) = \frac{v+\alpha}{1+\alpha}$.[Vetsikas and Jennings 2007]

[7] Let the system of differential equations be of size $\lambda \times \lambda$.

[8] Based on related ongoing work that we have been conducting, we note that systems of this form are generated also in other problems, like e.g. in the scenario where bidders can have multi-unit demand.

(1) As these equations hold $\forall v_i \in [v_l, v_h]$, we can substitute in each equation v_i with a new variable $z_i : z_i = g_{\alpha_j}(v_i), \forall z_i \in [g_{\alpha_j}(v_l), g_{\alpha_j}(v_h)]$. Note that α_j is equal to that which generated the particular equation from the general formula. E.g. in equation 31 of the example of the next section, we make the substitution $z_i = g_1(v_i)$.

(2) In the new equations the derivatives $g'_{\alpha_j}()$ appear in the form $\frac{1}{g'_{\alpha_j}(g^{-1}_{\alpha_j}(z_i))}$. The system of differential equations is linear in these variables and therefore, we can solve it to get equations, each one of which only has one of the derivatives of the bidding functions $g'_{\alpha_j}()$.

(3) Now we once more change the variables of the new equations using $y_i = g^{-1}_{\alpha_j}(z_i)$; we do each transformation in the corresponding equation that computes the derivative $g'_{\alpha_j}()$.

(4) The final system is in the form $g'_{\alpha_j}(y_i) = \mathcal{F}_{\alpha_j}\Big(y_i, g_{\alpha_j}(y_i), \underbrace{g^{-1}_{\alpha_k}(g_{\alpha_j}(y_i))}_{\forall k \neq j}\Big)$, where $\mathcal{F}_{\alpha_j}()$ is some function of the variables y_i, $g_{\alpha_j}(y_i)$ and $g^{-1}_{\alpha_k}(g_{\alpha_j}(y_i)), \forall k \neq j$. The boundary conditions are $g_{\alpha_j}(R) = R$.[9] Additionally the minimum and maximum values obtained by each variable y_i are the same as those of v_i. Let us define the maximum attainable value as $\tilde{R}$. Now this system resembles vaguely the form solved by the Euler method. Inspired by this, we modify the Euler method and propose the following algorithm for solving this system:[10]

(a) Initialization: Set $x_j = R, \forall j$ and $g_j(R) = R$.

(b) While not done do $\Big\{$

(c) choose $j : j = \arg\min g_j(x_j)$ and $x_j < \tilde{R}$

(d) $g_j(x_j + h) = g_j(x_j) + h \cdot \mathcal{F}_{\alpha_j}\big(x_j, g_j(x_j), \underbrace{g_k^{-1}(g_j(x_j))}_{\forall k \neq j}\big)$

(e) $x_j = x_j + h\Big\}$

This algorithm works because in line (d), we can use a linear interpolation, using the values of $g_k()$ that we have computed previously in the algorithm, to find the value of any $g_k^{-1}(g_j(x_j)), \forall k \neq j$. Note that we do indeed have all the necessary values of $g_k()$, and this is due to the way that we select each j in line (c).

5.3 Example 2: A General Example of Asymmetric Competitiveness

We will use a specific example, that of an m^{th} price auction, with $N = 3$ bidders and $m = 2$ items for sale, where there are two possible models, one where $\alpha = 0$ and $\alpha = 0.5$, both with probability 50%, to demonstrate the method (algorithm) presented in the previous section. In this example we have the following system of differential equations (obtained from equation 25 by setting $\alpha_i = 0, 0.5$ for probabilities $h(0) = h(0.5) = 0.5$):

$$g_0^{-1}(g_0(v_i)) + g_{0.5}^{-1}(g_0(v_i)) = \frac{v_i - g_0(v_i)}{g'_0(g_0^{-1}(g_0(v_i)))} + \frac{v_i - g_0(v_i)}{g'_{0.5}(g_{0.5}^{-1}(g_0(v_i)))} \tag{30}$$

$$0 = \frac{v_i - g_{0.5}(v_i) - 0.5v_i + 0.5g_0^{-1}(g_{0.5}(v_i))}{g'_0(g_0^{-1}(g_{0.5}(v_i)))} + \frac{v_i - g_{0.5}(v_i) - 0.5v_i + 0.5g_{0.5}^{-1}(g_{0.5}(v_i))}{g'_{0.5}(g_{0.5}^{-1}(g_{0.5}(v_i)))} \tag{31}$$

Note that these hold $\forall v \in \{v_l, v_h\}$.

Now we will use the methodology and the algorithm of the previous section in order to solve the system of differential equations. First, we apply the transformation described by step (1) in

[9] We need to have the boundary condition defined at the same point R for all α_j.

[10] Note that this algorithm is given for a positive step size $h > 0$. In case the boundary condition is such that we need to use a negative step size, we need to change the loop to select the j with the highest (rather than the lowest) argument $g_{\alpha_j}(x_j)$. Additionally, $\tilde{R}$ now represents the lowest value attained by y_i (and v_i).

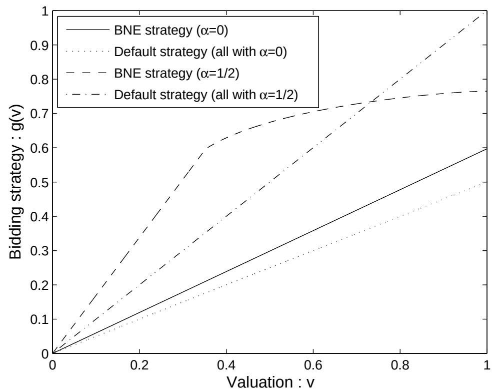

Fig. 1. Equilibrium strategies $g(v)$ for an m^{th} price auction in which there is a 50% chance that the opponents are strictly self-interested ($\alpha = 0$) and a 50% chance that they are competitive with $\alpha = \frac{1}{2}$. The equilibrium strategy is different depending on whether the bidder has $\alpha = 0$ or $\alpha = \frac{1}{2}$. Also included are the default equilibrium strategies (all the opponents have the same competition factor α as the bidder). The valuations v are drawn from the uniform distribution $U[0, 1]$. The number of bidders is $N = 3$ and the number of items being sold is $m = 2$.

order to get:

$$g_0^{-1}(z_i) + g_{0.5}^{-1}(z_i) = \frac{g_0^{-1}(z_i) - z_i}{g_0'(g_0^{-1}(z_i))} + \frac{g_0^{-1}(z_i) - z_i}{g_{0.5}'(g_{0.5}^{-1}(z_i))} \tag{32}$$

$$0 = \frac{0.5g_{0.5}^{-1}(z_i) + 0.5g_0^{-1}(z_i) - z_i}{g_0'(g_0^{-1}(z_i))} + \frac{g_{0.5}^{-1}(z_i) - z_i}{g_{0.5}'(g_{0.5}^{-1}(z_i))} \tag{33}$$

Solving the linear system as described by step (2) yields:

$$g_0'(g_0^{-1}(z_i)) = \frac{1}{2}\frac{g_0^{-1}(z_i) - z_i}{g_{0.5}^{-1}(z_i) + g_0^{-1}(z_i)}\frac{g_0^{-1}(z_i) - g_{0.5}^{-1}(z_i)}{z_i - g_{0.5}^{-1}(z_i)} \tag{34}$$

$$g_{0.5}'(g_{0.5}^{-1}(z_i)) = \frac{1}{2}\frac{g_0^{-1}(z_i) - z_i}{g_{0.5}^{-1}(z_i) + g_0^{-1}(z_i)}\frac{g_0^{-1}(z_i) - g_{0.5}^{-1}(z_i)}{0.5g_{0.5}^{-1}(z_i) + 0.5g_0^{-1}(z_i) - z_i} \tag{35}$$

After this, we get the final system by applying the transformation described by step (3):

$$g_0'(y_i) = \frac{1}{2}\frac{y_i - g_0(y_i)}{g_{0.5}^{-1}(g_0(y_i)) + y_i}\frac{y_i - g_{0.5}^{-1}(g_0(y_i))}{g_0(y_i) - g_{0.5}^{-1}(g_0(y_i))} \tag{36}$$

$$g_{0.5}'(y_i) = \frac{1}{2}\frac{g_0^{-1}(g_{0.5}(y_i)) - g_{0.5}(y_i)}{y_i + g_0^{-1}(g_{0.5}(y_i))}\frac{g_0^{-1}(g_{0.5}(y_i)) - y_i}{0.5y_i + 0.5g_0^{-1}(g_{0.5}(y_i)) - g_{0.5}(y_i)} \tag{37}$$

Now these equations give the functions $\mathcal{F}_j()$ used in the algorithm. After running the algorithm we obtain the solution graphed in figure 1.[11] Examining the solution, we notice that both types of bidders actually bid more than they would in the case when they faced bidders of the same type as them. The self-interested agents ($\alpha = 0$) bid more than in the case when they

[11] We would like to point out that the algorithm is quite sensitive (i.e. not stable) to the initial points set in it. Unfortunately, it is a common problem when solving numerically the differential equations that describe equilibria in pretty much any auction scenario, that the boundary conditions at the start of the computation need to be moved slightly from those bounds in order to be able to run the differential equation solvers. In most other auction settings though, this is not a problem as the process is fairly stable, however, in this setting, it is not, and thus, depending on the initial conditions set in it, we've noticed that the end result could change somewhat. This is an issue that we are currently trying to resolve.

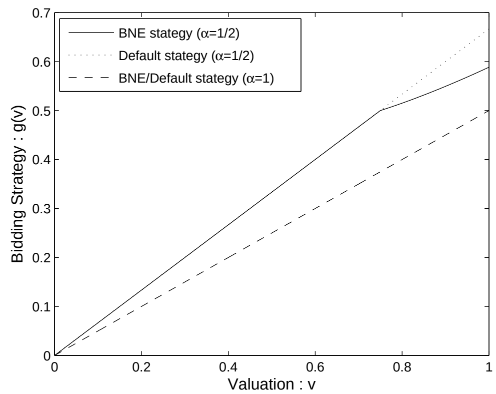

Fig. 2. Equilibrium strategies $g(v)$ for an m^{th} price auction in which bidders use the CRRA utility function $u(x) = x^\alpha$ and there is a 50% chance that the opponents are risk neutral ($\alpha = 1$) and a 50% chance that they are risk averse with $\alpha = \frac{1}{2}$. Also included are the default equilibrium strategies (all the opponents have the same competition factor α as the bidder); in the case when $\alpha = 1$, the default strategy is the same of the BNE strategy, so it is not graphed separately. The valuations v are drawn from the uniform distribution $U[0, 1]$. The number of bidders is $N = 3$ and the number of items being sold is $m = 2$.

only face other self-interested agents, because some of the opponents now bid higher. On the other hand, the competitive agents ($\alpha = \frac{1}{2}$) also bid more than before, because now some of the opponents are self-interested and bid lower; however, once they reach the maximum bid that the self-interested opponents will ever place, as they only have to face fewer opponents all with $\alpha = \frac{1}{2}$ beyond that point, the rate of bid increase is significantly decreased.

5.4 Example 3: Asymmetric Risk Attitudes

We now apply our methodology for solving these systems to the system of corollary 1. By applying the first transformation (step 1), we get $\forall z_i, \alpha_i$ that:

$$(N-m) \sum_{\alpha \in X_H} \frac{F'(g_\alpha^{-1}(z_i))}{g'_\alpha(g_\alpha^{-1}(z_i))} h(\alpha) = \frac{u'_{\alpha_i}(g_{\alpha_i}^{-1}(z_i) - z_i)}{u_{\alpha_i}(g_{\alpha_i}^{-1}(z_i) - z_i) - u_{\alpha_i}(0)} \sum_{\alpha \in X_H} F(g_\alpha^{-1}(z_i)) h(\alpha) \tag{38}$$

In order for this system to have a solution it must be:

$$\frac{u'_{\alpha_1}(g_{\alpha_1}^{-1}(z) - z)}{u_{\alpha_1}(g_{\alpha_1}^{-1}(z) - z) - u_{\alpha_i}(0)} = \ldots = \frac{u'_{\alpha_\lambda}(g_{\alpha_\lambda}^{-1}(z) - z)}{u_{\alpha_\lambda}(g_{\alpha_\lambda}^{-1}(z) - z) - u_{\alpha_\lambda}(0)} \tag{39}$$

This gives $(\lambda - 1)$ independent equations which we can use to substitute all $g'_{\alpha_i}()$ with one of them, i.e. $g'_{\alpha_1}()$ using the following formula which is derived from equation 39:

$$\frac{1}{g'_{\alpha_i}(g_{\alpha_i}^{-1}(z))} = 1 + (\frac{1}{g'_{\alpha_1}(g_{\alpha_1}^{-1}(z))} - 1) \cdot \tag{40}$$

$$\frac{u'_{\alpha_i}(g_{\alpha_i}^{-1}(z) - z) u'_{\alpha_1}(g_{\alpha_1}^{-1}(z) - z) - u''_{\alpha_1}(g_{\alpha_1}^{-1}(z) - z)(u_{\alpha_i}(g_{\alpha_i}^{-1}(z) - z) - u_{\alpha_i}(0))}{u'_{\alpha_1}(g_{\alpha_1}^{-1}(z) - z) u'_{\alpha_i}(g_{\alpha_i}^{-1}(z) - z) - u''_{\alpha_i}(g_{\alpha_i}^{-1}(z) - z)(u_{\alpha_1}(g_{\alpha_1}^{-1}(z) - z) - u_{\alpha_1}(0))}$$

Now, in equation 38, for the specific value $\alpha_i = \alpha_1$, we substitute the terms $\frac{1}{g'_{\alpha_i}(g_{\alpha_i}^{-1}(z))}$ using equation 40, for all values $i = 2, \ldots, \lambda$. Thus the final differential equation only contains term $\frac{1}{g'_{\alpha_1}(g_{\alpha_1}^{-1}(z))}$. By applying step 3 of the methodology and making the substitution $y = g_{\alpha_1}^{-1}(z)$, the final differential equation computes term $g'_{\alpha_1}(y)$. This term's computation (see step (d) of our algorithm) in this differential equation uses the values of $g_{\alpha_i}^{-1}(g_{\alpha_1}(y))$. These values are derived from this equation (which follows from equation 39 when we make the substitution $y = g_{\alpha_1}^{-1}(z)$):

$$\frac{u'_{\alpha_i}(g_{\alpha_i}^{-1}(g_{\alpha_1}(y)) - g_{\alpha_1}(y))}{u_{\alpha_i}(g_{\alpha_i}^{-1}(g_{\alpha_1}(y)) - g_{\alpha_1}(y)) - u_{\alpha_i}(0)} = \frac{u'_{\alpha_1}(y - g_{\alpha_1}(y))}{u_{\alpha_1}(y - g_{\alpha_1}(y)) - u_{\alpha_i}(0)} \tag{41}$$

Since y and $g_{\alpha_1}(y)$ are known we can solve this equation to get the value of $g_{\alpha_i}^{-1}(g_{\alpha_1}(y))$. We used the Bisection Method to accomplish this, which is slow but safe in that a solution is always found; we take the algorithm from chapter 9 of [Press et al. 2007], where there are many other methods that could be alternatively used.

We give now a simple example of asymmetric risk attitudes. We examine an m^{th} price auction, with $N = 3$ bidders and $m = 2$ items for sale, where there are two possible models of bidders using the CRRA utility function $u_\alpha(x) = x^\alpha$, one where $\alpha = 1$ (risk neutral bidder) and another where $\alpha = 0.5$ (risk averse), both with probability 50%. In this example we have the following system of equations (obtained from equations 39 and 38 by setting $u_\alpha(x) = x^\alpha$ for $\alpha_i = 0.5, 1$ and probabilities $h(0.5) = h(1) = 0.5$):

$$\frac{1}{g_1^{-1}(z) - z} = \frac{0.5}{g_{0.5}^{-1}(z) - z} \tag{42}$$

$$\frac{1}{g_1'(g_1^{-1}(z))} + \frac{1}{g_{0.5}'(g_{0.5}^{-1}(z))} = \frac{0.5}{g_{0.5}^{-1}(z) - z}(g_{0.5}^{-1}(z) + g_1^{-1}(z)) \tag{43}$$

We present the equilibrium strategies in figure 2. It is interesting to note that the strategy for risk-neutral bidders is, in this example, identical to the case when all the bidders are risk-neutral. A similar effect is true for the risk-averse bidders as well, in that for most valuations, the bids are identical; however, when the valuation is high enough that the risk-neutral opponents would never outbid the risk-averse bidders, the latter increase their bids at a much lower rate as the valuation increases.

6. CONCLUSIONS

In this paper, we examined asymmetric bidder models both in risk attitudes and competitiveness. More specifically, we assumed that each bidder only knows its own model, which is how competitive he is or which is his utility function, and that the opponents use a variety of models (for their risk attitudes or their competitiveness) each with a certain probability. We gave a theoretical analysis and derived the systems of differential equations that characterize the Bayes-Nash equilibria that exist in both these cases. This was the primary contribution of the paper. As these systems of differential equations are quite complicated, we also provided a methodology (and an algorithm) for solving these systems, which constitutes our second contribution.

There are still a number of issues we are currently pursuing. First, we are trying to improve the stability of the solutions we generate, which is still an issue with our algorithm. Furthermore, we are currently pursuing research into other auction issues, e.g. bidders with multi-unit demand, where the same methodology and algorithm (albeit with some extensions) to those we presented here seem to be able to generate solutions to those problems as well.

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