A New Encoding and Implementation of Not Necessarily Closed Convex Polyhedra*

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Abstract. Convex polyhedra, commonly employed for the analysis and verification of both hardware and software, may be defined either by a finite set of linear inequality constraints or by finite sets of generating points and rays of the polyhedron. Although most implementations of the polyhedral operations assume that the polyhedra are topologically closed (i.e., all the constraints defining them are non-strict), several analyzers and verifiers need to compute on a domain of convex polyhedra that are not necessarily closed (NNC). The usual approach to implementing NNC polyhedra is to embed them into closed polyhedra in a vector space having one extra dimension and reuse the tools and techniques already available for closed polyhedra. Previously, this embedding has been designed so that a constant number of constraints and a linear number of generators have to be added to the original NNC specification of the polyhedron. In this paper we explore an alternative approach: while still using an extra dimension to represent the NNC polyhedron by a closed polyhedron, the new embedding adds a linear number of constraints and a constant number of generators. As far as the issue of providing a non-redundant description of the NNC polyhedron is concerned, we generalize the results established in a previous paper so that they apply to both encodings.

1 Introduction

Many applications of static analysis and verification compute on some abstract domain based on convex polyhedra [6]. Traditionally, most of these applications are restricted to convex polyhedra that are topologically closed. When adopting the *Double Description* (DD) method [11], a closed convex polyhedron can be specified in two ways, using a *constraint system* or a *generator system*: the constraint system contains a finite set of linear non-strict inequality constraints; the generator system contains two finite sets of vectors, collectively called *generators*, which are rays and points of the polyhedron.

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Some applications of static analysis and verification, including recent proposals such as [5], need to compute on the domain of *not necessarily closed* (NNC) convex polyhedra. By definition, any NNC polyhedron can be represented by a so-called *mixed constraint system*, that is, a constraint system where a further finite set of linear *strict* inequality constraints is allowed to occur. The usual approach for implementing NNC polyhedra is to embed them into closed polyhedra in a vector space with one extra dimension. While this idea, originally proposed in [8] and also described in [9], proved to be quite effective, its direct application results in a low-level user interface where most of the geometric intuition of the DD method gets lost under the "implementation details".[3]

A much cleaner approach was proposed in [1, 3], where the concept of generator of an NNC polyhedron is extended to also account for the *closure points* of the polyhedron. In particular, it is shown that any NNC polyhedron can be defined directly by means of an *extended generator system*, namely, a triple of finite sets containing rays, points and closure points of the polyhedron. By combining the mixed constraint systems with these extended generator systems for describing NNC polyhedra we can obtain a two-fold improvement over the proposal in [8, 9]: easier generalizations and a natural, implementation-independent interface.

Easier generalizations. Several complex operators, whose definition is in terms of the rays and points of the standard generator systems for closed polyhedra, need to be generalized to NNC polyhedra. Examples are given by the *time-elapse* operator of [8, 9] and the generators-based widening of [4]. The notion of extended generator system proved to be very effective in the definition and justification of these generalizations. As an example, let us consider a very basic operator: the inclusion test between two polyhedra. The usual implementation for closed polyhedra is based on the following specification in terms of their constraint and generator systems. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be closed polyhedra such that $\mathcal{P}_1$ is defined by the generator system $\mathcal{G}_1$ and $\mathcal{P}_2$ by the constraint system $\mathcal{C}_2$. Then we have $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if all the generators in $\mathcal{G}_1$ *satisfy* all the constraints in $\mathcal{C}_2$. In order to test whether or not a generator $\boldsymbol{g}$ satisfies a constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$, it is sufficient to determine if the scalar product $s = \langle \boldsymbol{a}, \boldsymbol{g} \rangle$ is such that $s \geq b$, when $\boldsymbol{g}$ is a point, or such that $s \geq 0$, when $\boldsymbol{g}$ is a ray. Consider now the generalization to two polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$ that are not necessarily closed. With the high-level interface proposed in [3], the inclusion test can be easily specified using *the same approach* described above: we only need to generalize the case analysis of the satisfaction test to also cover the combinations provided by the additional constraint and generator types (i.e., strict inequalities and closure points), as shown in Table 1. The elegance of this generalization is better appreciated if contrasted with the specification of the inclusion test on the low-level implementation of [8], informally described in the same paper, which appears to be much more tricky

[3] This has a direct, negative impact on the usability of the resulting software: on this subject, see [3, Section 4.1, page 218], [7, Section 4.5, pp. 10–11], and [10, Section 1.1.4, page 10].

and obscure. The reason is that in [8] the reader has no high-level interpretation of the generators occurring in the low-level encoding.

	Generator type		
Constraint type	ray	point	closure point
non-strict inequality	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq 0$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq b$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq b$
strict inequality	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq 0$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle > b$	$\langle \boldsymbol{a}, \boldsymbol{g} \rangle \geq b$

Table 1. Checking whether a constraint is satisfied by a generator.

A natural, implementation-independent interface. The combination of mixed constraint systems and extended generator systems offers another improvement over the proposal in [8, 9]: a high-level user interface that is completely separate from the implementation. On the one hand, an NNC polyhedron can be presented to the client application directly in terms of its defining strict and non-strict constraints or its generating rays, points and closure points; there is no need for the client to be aware of the use of an additional space dimension in the implementation and all issues related to its correct handling, such the strong minimization procedures [3]. On the other hand, by relying on the high-level specification only, the client application will be unaffected by the wider adoption of lazy and incremental computation techniques in the procedures implementing the operators on convex polyhedra. Moreover, if all the functionalities and invariants of the interface are maintained, it is then possible to change the low-level data structures without affecting the application.

In this paper we exploit the latter possibility by introducing an alternative class of closed polyhedra for implementing the NNC polyhedra. The basis of this representation is a simple generalization of the class of polyhedra used in [8, 9] and also in [3]. The new class continues to employ an additional dimension to encode whether or not each affine half-space defining the NNC polyhedron is closed and relies on the same semantic function given in [3] for extracting the NNC polyhedron it embeds. We describe two alternative specializations of this class for representing the NNC polyhedra. One of these, shown to be biased for the use of the constraint representation, corresponds to the embedding defined in [3] while the other, which is biased for the use of the generator representation, is new to this paper. Moreover, we generalize the notion of strong minimal form [3] so that it is applicable to all the above classes of closed polyhedra.

One interesting and potentially useful consequence of having the option of these alternative implementations is that, depending on the number of strict constraints in the constraint system compared with the number of closure points that are also points in the generator system, the choice of representation will affect the efficiency of the polyhedral operations. The *Parma Polyhedra Library*[4],

[4] Publicly available at URI `http://www.cs./unipr.it/ppl/`. The implementation described in this paper is available in the `alt_nnc` branch of the PPL's CVS repository.

a modern C++ library for the manipulation of convex polyhedra, has been extended so as to implement both approaches, so that it will be possible to perform experiments to compare their efficiencies.

The paper is structured as follows: Section 2 recalls the required concepts and notations; Section 3 presents a general class and two special subclasses of the set of closed polyhedra that are appropriate for the representation of NNC polyhedra; Section 4 generalizes, to all the above classes of closed polyhedra, the notion of strong minimal form introduced in [3]; Section 5 concludes. The reader is referred to [2] for the proofs of all the stated results.

2 Preliminaries

We first define some necessary terminology and notation.

The set of non-negative reals is denoted by $\mathbb{R}_+$. In the paper, all topological arguments refer to the Euclidean topological space $\mathbb{R}^n$, for $n \in \mathbb{N}$. If $S \subseteq \mathbb{R}^n$, then the *topological closure* $\mathbb{C}(S)$ is defined as $\bigcap\{\, C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed} \,\}$.

For each $i \in \{1, \ldots, n\}$, v_i denotes the i-th component of the (column) vector $\boldsymbol{v} \in \mathbb{R}^n$. We denote by $\mathbf{0}$ the vector of $\mathbb{R}^n$ having all components equal to zero. A vector $\boldsymbol{v} \in \mathbb{R}^n$ can also be interpreted as a matrix in $\mathbb{R}^{n\times 1}$ and manipulated accordingly with the usual definitions for addition, multiplication (both by a scalar and by another matrix), and transposition, which is denoted by $\boldsymbol{v}^{\mathrm{T}}$. The *scalar product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, denoted $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$, is the real number $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \sum_{i=1}^{n} v_i w_i$.

For any relational operator $\bowtie \in \{=, \geq, \leq, <, >\}$, we write $\boldsymbol{v} \bowtie \boldsymbol{w}$ to denote the conjunctive proposition $\bigwedge_{i=1}^{n}(v_i \bowtie w_i)$. In contrast, $\boldsymbol{v} \neq \boldsymbol{w}$ will denote the proposition $\neg(\boldsymbol{v} = \boldsymbol{w})$. For each vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \mathbf{0}$, the linear inequality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b$) defines a topologically closed (resp., open) affine half-space of $\mathbb{R}^n$. We do not distinguish between syntactically different constraints defining the same affine half-space so that, e.g., $x \geq 2$ and $2x \geq 4$ are considered to be the same constraint.

A subset $\mathcal{P}$ of $\mathbb{R}^n$ is called a *closed polyhedron* if either $\mathcal{P}$ can be expressed as the intersection of a finite number of closed affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all closed polyhedra on $\mathbb{R}^n$ is denoted by $\mathbb{CP}_n$. A subset $\mathcal{P}$ of $\mathbb{R}^n$ is called an *NNC polyhedron* if either $\mathcal{P}$ can be expressed as the intersection of a finite number of (not necessarily closed) affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all NNC polyhedra on $\mathbb{R}^n$ is denoted by $\mathbb{P}_n$. The set $\mathbb{P}_n$, when partially ordered by subset inclusion, is a lattice and $\mathbb{CP}_n$ is a sublattice of $\mathbb{P}_n$ (note that $\mathbb{CP}_n = \mathbb{P}_n$ if and only if $n = 0$). The binary meet operation is given by set-intersection, whereas the binary join operation, denoted $\uplus$, is called *convex polyhedral hull*, *poly-hull* for short. In this paper, we only consider polyhedra in $\mathbb{P}_n$ when $n > 0$.

A *mixed constraint system* $\mathcal{C}$ is a finite set of linear inequality constraints and we write $\mathrm{con}(\mathcal{C})$ to denote the polyhedron described by $\mathcal{C}$.

Suppose that $\mathcal{P} \in \mathbb{P}_n$ is non-empty. A vector $\boldsymbol{p} \in \mathbb{R}^n$ is a *point* of $\mathcal{P}$ if $\boldsymbol{p} \in \mathcal{P}$; a vector $\boldsymbol{r} \in \mathbb{R}^n$ such that $\boldsymbol{r} \neq \mathbf{0}$ is a *ray* of $\mathcal{P}$ if, for every point $\boldsymbol{p} \in \mathcal{P}$ and every $\rho \in \mathbb{R}_+$, we have $\boldsymbol{p} + \rho\boldsymbol{r} \in \mathcal{P}$; a vector $\boldsymbol{c} \in \mathbb{R}^n$ is a *closure point* of $\mathcal{P}$ if $\boldsymbol{c} \in \mathbb{C}(\mathcal{P})$.

Given three finite sets of vectors $R, P, C \subseteq \mathbb{R}^n$, where $R = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r\}$ and $\mathbf{0} \notin R$, $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p\}$ and $C = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_c\}$, then the triple $\mathcal{G} = (R, P, C)$ is called an *extended generator system* [3] for the NNC polyhedron

$$\mathrm{gen}(\mathcal{G}) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{r} \rho_i \boldsymbol{r}_i + \sum_{i=1}^{p} \pi_i \boldsymbol{p}_i + \sum_{i=1}^{c} \gamma_i \boldsymbol{c}_i \;\middle|\; \begin{array}{l} \boldsymbol{\rho} \in \mathbb{R}^r_+, \boldsymbol{\pi} \in \mathbb{R}^p_+, \boldsymbol{\gamma} \in \mathbb{R}^c_+, \\ \boldsymbol{\pi} \neq \mathbf{0}, \sum_{i=1}^{p} \pi_i + \sum_{i=1}^{c} \gamma_i = 1 \end{array} \right\}.$$

The polyhedron $\mathrm{gen}(\mathcal{G})$ is empty if and only if $P = \varnothing$. For a non-empty polyhedron $\mathcal{P}$, vectors in R, P, and C are rays, points and closure points of $\mathcal{P}$, respectively. We define an ordering $\sqsubseteq$ on extended generator systems such that, for any generator systems $\mathcal{G}_1 = (R_1, P_1, C_1)$ and $\mathcal{G}_2 = (R_2, P_2, C_2)$, $\mathcal{G}_1 \sqsubseteq \mathcal{G}_2$ if and only if $R_1 \subseteq R_2$, $P_1 \subseteq P_2$ and $C_1 \subseteq C_2$; if, in addition, $\mathcal{G}_1 \neq \mathcal{G}_2$, we write $\mathcal{G}_1 \sqsubset \mathcal{G}_2$. When $C = \varnothing$, we will omit it from the generator system and simply write $\mathcal{G} = (R, P)$. In this case, the system $\mathcal{G}$ that defines the closed polyhedron $\mathcal{P} = \mathrm{gen}(\mathcal{G})$, is called a *(standard) generator system* for $\mathcal{P}$.

Consider a mixed constraint system $\mathcal{C}$, an extended generator system $\mathcal{G}$, and a polyhedron $\mathcal{P}$. If $\mathrm{con}(\mathcal{C}) = \mathrm{gen}(\mathcal{G}) = \mathcal{P}$, then $(\mathcal{C}, \mathcal{G})$ is said to be a *DD pair* for $\mathcal{P}$, and we write $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. We say that

- $\mathcal{C}$ is in *minimal form* if there does not exist $\mathcal{C}' \subset \mathcal{C}$ such that $\mathrm{con}(\mathcal{C}') = \mathcal{P}$;
- $\mathcal{G}$ is in *minimal form* if there does not exist $\mathcal{G}' \sqsubset \mathcal{G}$ such that $\mathrm{con}(\mathcal{G}') = \mathcal{P}$;
- the DD pair $(\mathcal{C}, \mathcal{G})$ is in *minimal form* if $\mathcal{C}$ and $\mathcal{G}$ are both in minimal form.

3 Representing NNC Polyhedra

The idea underlying the proposal of [8, 9] is to encode each NNC polyhedron of $\mathbb{P}_n$ into a closed polyhedron of $\mathbb{CP}_{n+1}$. In the following, we denote by ϵ the variable corresponding to the $(n+1)$-st Cartesian axis of $\mathbb{R}^{n+1}$. The interpretation function $[\![\cdot]\!] \colon \mathbb{CP}_{n+1} \to \mathbb{P}_n$ maps any closed polyhedron in $\mathbb{CP}_{n+1}$ to an NNC polyhedron in $\mathbb{P}_n$; in particular, points in the closed polyhedron with a positive ϵ-coordinate correspond to points in the NNC polyhedron.

Definition 1. (Represented NNC polyhedron.) *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be a closed polyhedron. $\mathcal{R}$ is said to* represent *the NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ if and only if*

$$\mathcal{P} = [\![\mathcal{R}]\!] \stackrel{\text{def}}{=} \left\{ \boldsymbol{v} \in \mathbb{R}^n \;\middle|\; \exists e \in \mathbb{R} \,.\, \big(e > 0 \wedge (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R}\big) \right\}. \tag{1}$$

Note that any closed polyhedron that is included in the half-space defined by the constraint $\epsilon \leq 0$ actually represents the empty NNC polyhedron.

Not all the polyhedra in $\mathbb{CP}_{n+1}$ are good candidates for representing an NNC polyhedron in $\mathbb{P}_n$. The rationale driving the choice of an appropriate subclass of $\mathbb{CP}_{n+1}$ is that most of the operators defined on the domain of closed polyhedra could be used, with no more than minor modifications, to implement corresponding operators on the domain of NNC polyhedra. For instance, one would like to implement the intersection and the poly-hull of two NNC polyhedra by computing the intersection and the poly-hull of their closed representations, respectively.

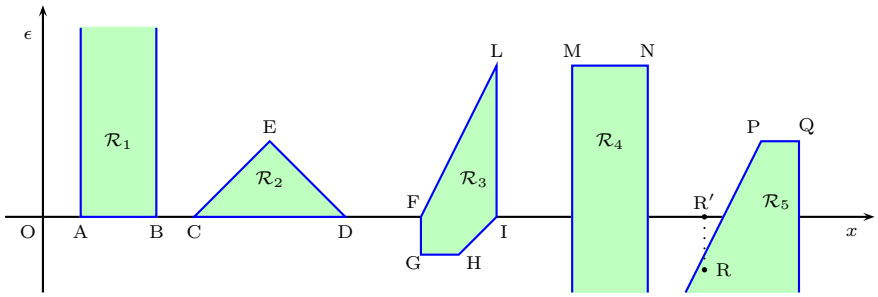

Fig. 1. Only $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ are ϵ-polyhedra.

Under such a requirement, we will define two alternative representations for NNC polyhedra. The two classes of closed polyhedra used for these representations are instances of a more general class of closed polyhedra.

Definition 2. (ϵ-polyhedron.) *A closed polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be an* ϵ-polyhedron *if and only if*

$$\exists \delta \in \mathbb{R} \,.\, \Big(\delta > 0 \wedge \mathcal{R} \subseteq \mathrm{con}\big(\{\epsilon \leq \delta\}\big)\Big); \tag{2}$$

$$\forall \boldsymbol{v} \in \mathbb{R}^n, e \in \mathbb{R} : (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in \mathcal{R} \implies (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathcal{R}. \tag{3}$$

The polyhedron $\mathcal{R}$ is said to be an ϵ-polyhedron for $\mathcal{P} \in \mathbb{P}_n$, *denoted $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$, if $\mathcal{R}$ is an ϵ-polyhedron and $\mathcal{P} = [\![\mathcal{R}]\!]$.*

Condition (3) that every point in the ϵ-polyhedron $\mathcal{R}$ has a projection on the hyperplane defined by the constraint $(\epsilon = 0)$ corresponds to a dual property concerning the constraints for $\mathcal{R}$.

Proposition 1. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ be such that $\mathcal{R} \subseteq \mathrm{con}\big(\{\epsilon \leq \delta\}\big)$, where $\delta > 0$. Then $\mathcal{R}$ is an ϵ-polyhedron if and only if*

$$\mathcal{R} \subseteq \mathrm{con}\Big(\big\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + s \cdot \epsilon \geq b\big\}\Big) \implies \mathcal{R} \subseteq \mathrm{con}\Big(\big\{\langle \boldsymbol{a}, \boldsymbol{x}\rangle + 0 \cdot \epsilon \geq b\big\}\Big). \tag{4}$$

In Figure 1 we show several examples of polyhedra in $\mathbb{CP}_2$ (representing NNC polyhedra in $\mathbb{P}_1$), a subset of which happens to be ϵ-polyhedra. In particular, the semi-column polyhedron $\mathcal{R}_1$, which according to Definition 1 represents the closed interval $\mathcal{P}_1 = \mathrm{con}\big(\{1 \leq x \leq 3\}\big)$, is not an ϵ-polyhedron, because it is not provided with a finite upper-bound on the ϵ coordinate, therefore violating condition (2) of Definition 2. The triangle $\mathcal{R}_2$ is an ϵ-polyhedron for the open segment $\mathcal{P}_2 = \mathrm{con}\big(\{4 < x < 8\}\big)$. Polyhedron $\mathcal{R}_3$ is an ϵ-polyhedron for the segment $\mathcal{P}_3 = \mathrm{con}\big(\{10 < x \leq 12\}\big)$, which is neither closed nor open. Similarly, $\mathcal{R}_4$ is an ϵ-polyhedron for the closed segment $\mathcal{P}_4 = \mathrm{con}\big(\{14 \leq x \leq 16\}\big)$. Finally, polyhedron $\mathcal{R}_5$ represents the NNC polyhedron $\mathcal{P}_5 = \mathrm{con}\big(\{18 \leq x \leq 20\}\big)$, but it is not an ϵ-polyhedron because it violates condition (3) of Definition 2. For instance, even though $\mathrm{R} \in \mathcal{R}_5$, we have $\mathrm{R}' \notin \mathcal{R}_5$.

If we are to provide an implementation-independent interface for the user, we need to be able to extract from the constraint and generator systems describing an ϵ-polyhedron, the corresponding mixed constraint system and extended generator system describing the NNC polyhedron it represents. Reasoning at the intuitive level, consider an arbitrary ϵ-polyhedron, such as $\mathcal{R}_3$ in Figure 1. Then, it is worth noting that any facet that is parallel to the ϵ axis, such as the segment [I, L], corresponding to an inequality constraint having a zero coefficient for the ϵ variable, will encode a *non-strict* inequality constraint of the represented NNC polyhedron $\mathcal{P}_3$ (in this case, the constraint $x \leq 12$). On the other hand, any facet such as the segment [L, F], corresponding to an inequality constraint having a negative coefficient for the ϵ variable, will encode a *strict* inequality constraint of the represented NNC polyhedron $\mathcal{P}_3$ (in this case, the constraint $x > 10$). Equivalently, we could have noted that in polyhedron $\mathcal{R}_3$ the points having a strictly positive ϵ coordinate can be chosen arbitrarily close to vertex $\mathrm{F} = (10, 0)^{\mathrm{T}}$, but all the points having value 10 for their x coordinate happen to have a non-positive ϵ coordinate. Thus, the vector $\mathrm{F}' = (10) \in \mathbb{R}^1$ represented by F is not a point of the NNC polyhedron $\mathcal{P}_3$, but it is one of its closure points. All of the above observations can be formalized as follows.

Definition 3. (Encoded descriptions.) *Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be a DD pair for a closed polyhedron. Then, if $[\![\mathcal{R}]\!] \neq \varnothing$, the* mixed constraint system encoded by $\mathcal{C}$ *is defined as* $\mathrm{con\_enc}(\mathcal{C}) \stackrel{\text{def}}{=} \mathcal{C}_S \cup \mathcal{C}_{NS}$, *where*

$$\mathcal{C}_S \stackrel{\text{def}}{=} \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle > b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\big) \in \mathcal{C}, \boldsymbol{a} \neq \boldsymbol{0}, s < 0 \Big\},$$
$$\mathcal{C}_{NS} \stackrel{\text{def}}{=} \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\big) \in \mathcal{C}, \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\big) \notin \mathcal{C}_S \Big\}.$$

If $[\![\mathcal{R}]\!] = \varnothing$, then we define $\mathrm{con\_enc}(\mathcal{C}) \stackrel{\text{def}}{=} \{x_1 > 0, -x_1 > 0\}$. *Also, the* extended generator system encoded by $\mathcal{G} = (R, P)$ *is defined as* $\mathrm{gen\_enc}(\mathcal{G}) \stackrel{\text{def}}{=} (R', P', C')$, *where*

$$R' \stackrel{\text{def}}{=} \big\{ \boldsymbol{r} \;\big|\; (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \in R \big\},$$
$$P' \stackrel{\text{def}}{=} \big\{ \boldsymbol{p} \;\big|\; (\boldsymbol{p}^{\mathrm{T}}, e)^{\mathrm{T}} \in P, e > 0 \big\},$$
$$C' \stackrel{\text{def}}{=} \big\{ \boldsymbol{c} \;\big|\; (\boldsymbol{c}^{\mathrm{T}}, 0)^{\mathrm{T}} \in P, \boldsymbol{c} \notin P' \big\}.$$

The following proposition states the correctness of the two mappings introduced above.

Proposition 2. *Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$ be an ϵ-polyhedron. Then*

$$[\![\mathcal{R}]\!] = \mathrm{con}\big(\mathrm{con\_enc}(\mathcal{C})\big) = \mathrm{gen}\big(\mathrm{gen\_enc}(\mathcal{G})\big). \tag{5}$$

Figure 2, which shows the poly-hulls of some of the polyhedra in Figure 1, provides a graphical and informal justification for the two conditions stated in Definition 2. Suppose we do not enforce condition (2) of Definition 2, thus

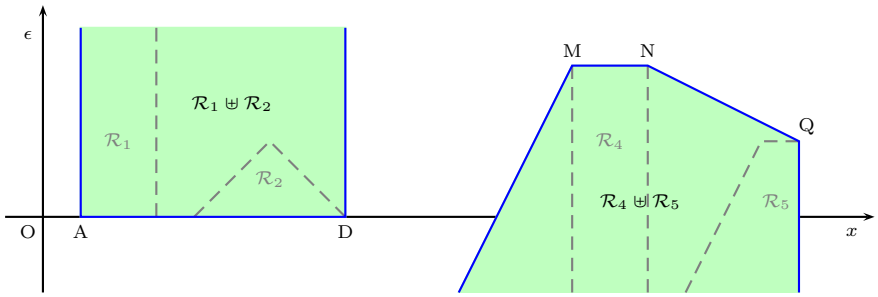

Fig. 2. $\mathcal{R}_1 \uplus \mathcal{R}_2$ does not represent the NNC polyhedron $\mathcal{P}_1 \uplus \mathcal{P}_2$; similarly, $\mathcal{R}_4 \uplus \mathcal{R}_5$ does not represent the NNC polyhedron $\mathcal{P}_4 \uplus \mathcal{P}_5$.

admitting polyhedra such as $\mathcal{R}_1$, and consider the poly-hull $\mathcal{P}_1 \uplus \mathcal{P}_2 = \mathrm{con}\big(\{1 \leq x < 8\}\big)$. The poly-hull $\mathcal{R}_1 \uplus \mathcal{R}_2$ of the two encodings for $\mathcal{P}_1$ and $\mathcal{P}_2$ represents a wrong result, since $[\![\mathcal{R}_1 \uplus \mathcal{R}_2]\!] = \mathrm{con}\big(\{1 \leq x \leq 8\}\big)$. Suppose now we do not enforce condition (3) of Definition 2, thus allowing for polyhedra such as $\mathcal{R}_5$, and consider the poly-hull $\mathcal{P}_4 \uplus \mathcal{P}_5 = \mathrm{con}\big(\{14 \leq x \leq 20\}\big)$. Again, the computation of this poly-hull using the closed encodings of its arguments provides a wrong result, since we have $[\![\mathcal{R}_4 \uplus \mathcal{R}_5]\!] = \mathrm{con}\big(\{12 < x \leq 20\}\big)$.

We now consider two special subclasses of the class of ϵ-polyhedra. The first of these requires zero as a lower bound for the ϵ dimension.

Definition 4. (C-ϵ-polyhedron.) *An ϵ-polyhedron $\mathcal{R} \in \mathbb{CP}_{n+1}$ is said to be* constraint-biased *and called a* C-ϵ-polyhedron *if and only if*

$$\mathcal{R} \subseteq \mathrm{con}\big(\{\epsilon \geq 0\}\big).$$

We write $\mathcal{R} \Rrightarrow_C \mathcal{P}$ if $\mathcal{R}$ is a C-ϵ-polyhedron and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$.

The set of constraint-biased ϵ-polyhedra corresponds, essentially, to the class of polyhedra originally proposed in [8, 9]. This is also the same class considered in [3], where these polyhedra were called ϵ-representations. Thus many of the definitions and results below concerning C-ϵ-polyhedra and the embedding of the NNC polyhedra in them are taken from [3].

In [3], we have shown how a C-ϵ-polyhedron for an NNC polyhedron $\mathcal{P}$ may be constructed directly from the constraint and generator systems for $\mathcal{P}$.

Definition 5. ($\mathrm{con\_repr}_C$ and $\mathrm{gen\_repr}_C$.) *Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. The* constraint-biased representation of $\mathcal{C}$ *is the constraint system* $\mathrm{con\_repr}_C(\mathcal{C})$ *on the vector space $\mathbb{R}^{n+1}$ where*

$$\begin{aligned}\mathrm{con\_repr}_C(\mathcal{C}) \stackrel{\text{def}}{=} {} & \{0 \leq \epsilon \leq 1\} \\ & \cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle - 1 \cdot \epsilon \geq b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\big) \in \mathcal{C} \Big\} \\ & \cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b\big) \in \mathcal{C} \Big\}.\end{aligned}$$

The constraint-biased representation of $\mathcal{G} = (R, P, C)$ *is the generator system* $\mathrm{gen\_repr}_C(\mathcal{G}) = (R', P')$ *on the vector space* $\mathbb{R}^{n+1}$ *where*

$$R' \stackrel{\text{def}}{=} \big\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\big|\; \boldsymbol{r} \in R \big\},$$
$$P' \stackrel{\text{def}}{=} \big\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}} \;\big|\; \boldsymbol{p} \in P \big\} \cup \big\{ (\boldsymbol{q}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\big|\; \boldsymbol{q} \in P \cup C \big\}.$$

Observe that, in the mapping defined by the representation function $\mathrm{gen\_repr}_C$ and using the notation in Definition 5, each point in P corresponds to two distinct points in P', having a positive and a zero ϵ coordinate, respectively. This ensures that condition (3) of Definition 2 is met. In general, the above encodings require a constant number of additional constraints versus a linear number of additional generators: this is the reason why ϵ-polyhedra in this subclass are called "constraint-biased".

The second special subclass of ϵ-polyhedra requires that all the non-empty ϵ-polyhedra have the ray $-\boldsymbol{e}_\epsilon \stackrel{\text{def}}{=} (\boldsymbol{0}^{\mathrm{T}}, -1)^{\mathrm{T}}$ so that there is no lower bound for the ϵ dimension.

Definition 6. (G-ϵ-polyhedron.) *An ϵ-polyhedron $\mathcal{R} = \mathrm{gen}\big((R, P)\big) \in \mathbb{CP}_{n+1}$ is said to be* generator-biased *and called a* G-ϵ-polyhedron *if and only if*

$$\mathcal{R} \supseteq \mathrm{gen}\Big(\big(\{-\boldsymbol{e}_\epsilon\}, P\big)\Big).$$

We write $\mathcal{R} \Rrightarrow_G \mathcal{P}$ if $\mathcal{R}$ is a G-ϵ-polyhedron and $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$.

As for the constraint-biased case, generator-biased ϵ-polyhedra can also be used for representing any NNC polyhedron. In particular, a G-ϵ-polyhedron for an NNC polyhedron $\mathcal{P}$ may be constructed directly from the constraint and generator systems for $\mathcal{P}$.

Definition 7. ($\mathrm{con\_repr}_G$ and $\mathrm{gen\_repr}_G$.) *Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron such that $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P}$. The* generator-biased representation of $\mathcal{C}$ *is the constraint system* $\mathrm{con\_repr}_G(\mathcal{C})$ *on the vector space* $\mathbb{R}^{n+1}$ *where*

$$\begin{aligned}
\mathrm{con_repr}_G(\mathcal{C}) \stackrel{\text{def}}{=} {} & \big\{0 \le \epsilon \le 1\big\} \\
& \cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle - 1 \cdot \epsilon \ge b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\big) \in \mathcal{C} \Big\} \\
& \cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \ge b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle > b\big) \in \mathcal{C} \Big\} \\
& \cup \Big\{ \langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \ge b \;\Big|\; \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \ge b\big) \in \mathcal{C} \Big\}.
\end{aligned}$$

The generator-biased representation of $\mathcal{G} = (R, P, C)$ *is the generator system* $\mathrm{gen\_repr}_G(\mathcal{G}) = (R', P')$ *on the vector space* $\mathbb{R}^{n+1}$ *where*

$$R' = \big\{-\boldsymbol{e}_\epsilon\big\} \cup \big\{ (\boldsymbol{r}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\big|\; \boldsymbol{r} \in R \big\},$$
$$P' = \big\{ (\boldsymbol{p}^{\mathrm{T}}, 1)^{\mathrm{T}} \;\big|\; \boldsymbol{p} \in P \big\} \cup \big\{ (\boldsymbol{q}^{\mathrm{T}}, 0)^{\mathrm{T}} \;\big|\; \boldsymbol{q} \in C \big\}.$$

It can be seen that, for each strict inequality contained in $\mathcal{C}$, the representation function $\mathrm{con\_repr}_G$ adds both the strict and the non-strict inequality encodings. This is similar to what is done for points in Definition 5 and, by virtue of Proposition 1, ensures that condition (3) of Definition 2 is met. In contrast, for each point in the generator system, the function $\mathrm{gen\_repr}_G$ does not add the corresponding closure point. In fact, these closure points are not needed, because they can be generated by combining the corresponding point with the ray $-\boldsymbol{e}_\epsilon$, which is always added. Since the encodings for ϵ-polyhedra in this subclass require a linear number of additional constraints versus a constant number of additional generators, they are called "generator-biased".

Returning to Figure 1, it can be observed that $\mathcal{R}_2$ is a constraint-biased ϵ-polyhedron, $\mathcal{R}_4$ is a generator-biased ϵ-polyhedron, whereas the ϵ-polyhedron $\mathcal{R}_3$ is neither constraint-biased nor generator-biased. By comparing $\mathcal{R}_3$ with $\mathcal{R}_1$ and $\mathcal{R}_2$ it can be seen that those ϵ-polyhedra that are not members of one of the two subclasses can require *both* a linear number of additional constraints and a linear number of additional generators (with respect to the original NNC descriptions), resulting in a significant waste of both memory space and computational time.

The following result formalizes the correctness of the encodings introduced in Definitions 5 and 7.

Proposition 3. *Let* $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{P} \in \mathbb{P}_n$. *Then*

1. $\mathrm{con}\big(\mathrm{con\_repr}_C(\mathcal{C})\big) \Rrightarrow_C \mathcal{P}$, $\quad \mathrm{con}\big(\mathrm{con\_repr}_G(\mathcal{C})\big) \Rrightarrow_G \mathcal{P}$;
2. $\mathrm{gen}\big(\mathrm{gen\_repr}_C(\mathcal{G})\big) \Rrightarrow_C \mathcal{P}$, $\quad \mathrm{gen}\big(\mathrm{gen\_repr}_G(\mathcal{G})\big) \Rrightarrow_G \mathcal{P}$.

The next proposition shows that most of the operators defined on the domain of NNC polyhedra $\mathbb{P}_n$ can be mapped into the corresponding operators on the class of ϵ-polyhedra defined on $\mathbb{CP}_{n+1}$.

Proposition 4. *Let* $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. *Suppose* $\mathcal{R} \Rrightarrow_Y \mathcal{P}$, *and* $\mathcal{R}_1 \Rrightarrow_Y \mathcal{P}_1$ *and* $\mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_2$. *Then*

1. $\mathcal{R}_1 \cap \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \cap \mathcal{P}_2$;
2. $(\mathcal{P}_1 \neq \varnothing \wedge \mathcal{P}_2 \neq \varnothing) \implies (\mathcal{R}_1 \uplus \mathcal{R}_2 \Rrightarrow_Y \mathcal{P}_1 \uplus \mathcal{P}_2)$;
3. *let* $f \stackrel{\text{def}}{=} \lambda \boldsymbol{x} \in \mathbb{R}^n \,.\, A\boldsymbol{x} + \boldsymbol{b}$ *be any affine transformation defined on* $\mathbb{P}_n$; *then* $g(\mathcal{R}) \Rrightarrow_Y f(\mathcal{P})$, *where*

$$g \stackrel{\text{def}}{=} \lambda \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} \in \mathbb{R}^{n+1} \,.\, \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \epsilon \end{pmatrix} + \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix}$$

is the corresponding affine transformation on $\mathbb{CP}_{n+1}$.

Hence, operations such as the intersection of NNC polyhedra and the application of affine transformations can be safely performed on any ϵ-polyhedra for the arguments; the same is true for the convex polyhedral hull operation, provided neither of the arguments is empty. Moreover, both the constraint-biased and the generator-biased subclasses are closed under the application of these operators.

4 Strong Minimization of ϵ-Polyhedra

As pointed out in [3], the usual minimization of the descriptions of a (constraint-biased) ϵ-polyhedron does not enforce the minimization of the encoded descriptions for the represented NNC polyhedron. The solution proposed in [3] is the definition of a stronger form of minimization to be applied to the descriptions of constraint-biased ϵ-polyhedra. We here define the same notion of strong minimal form, but this time for arbitrary ϵ-polyhedra, being careful that if it is constraint- or generator-biased before minimization, then it remains constraint- or generator-biased, respectively, after the minimization.

Definition 8. (Strong minimal form for ϵ-polyhedra.) *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P}$ and let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal form. Then*

- *$\mathcal{C}$ is in* strong minimal form *if there does not exist a constraint system $\mathcal{C}' \subset \mathcal{C}$ such that $\mathrm{con}\big(\mathcal{C}' \cup \{\epsilon \leq 1\}\big) \Rrightarrow_\epsilon \mathcal{P}$ and $\mathrm{con\_enc}(\mathcal{C}') \subset \mathrm{con\_enc}(\mathcal{C})$;*
- *$\mathcal{G}$ is in* strong minimal form *if there does not exist a generator system $\mathcal{G}' \sqsubset \mathcal{G}$ such that $\mathrm{gen}(\mathcal{G}') \Rrightarrow_\epsilon \mathcal{P}$ and $\mathrm{gen\_enc}(\mathcal{G}') \sqsubset \mathrm{gen\_enc}(\mathcal{G})$.*

The computation of strong minimal forms (smf's, for short) requires the removal of non-essential constraints and generators, whose efficient detection is based on the checking of particular saturation conditions. The following notation is needed for a formal definition of these conditions.

Let $\mathcal{R} = \mathrm{con}(\mathcal{C}) \in \mathbb{CP}_{n+1}$. The set of *strict* and *non-strict inequality encodings* $\mathcal{C}_>$ and $\mathcal{C}_\geq$ of $\mathcal{C}$ are defined as

$$\mathcal{C}_> \stackrel{\text{def}}{=} \Big\{ \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\big) \in \mathcal{C} \;\Big|\; \boldsymbol{a} \neq \boldsymbol{0}, s < 0 \Big\};$$
$$\mathcal{C}_\geq \stackrel{\text{def}}{=} \Big\{ \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\big) \in \mathcal{C} \;\Big|\; \boldsymbol{a} \neq \boldsymbol{0}, s = 0 \Big\}.$$

We say that a constraint $\big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + s \cdot \epsilon \geq b\big) \in \mathcal{C}$ is *unmatched* in $\mathcal{C}$ if $s < 0$ and $\big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\big) \notin \mathcal{C}$.

The sets of *point encodings* $\mathcal{G}_P \subseteq P$, *closure point encodings* $\mathcal{G}_C \subseteq P$, and *ray encodings* $\mathcal{G}_R \subseteq R$ of the generator system $\mathcal{G} = (R, P)$ are defined as follows:

$$\mathcal{G}_P \stackrel{\text{def}}{=} \big\{ (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \;\big|\; e > 0 \big\};$$
$$\mathcal{G}_C \stackrel{\text{def}}{=} \big\{ (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P \;\big|\; e = 0 \big\};$$
$$\mathcal{G}_R \stackrel{\text{def}}{=} \big\{ (\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in R \;\big|\; e = 0 \big\}.$$

A point $(\boldsymbol{v}^{\mathrm{T}}, e)^{\mathrm{T}} \in P$ is said to be *unmatched* in $\mathcal{G}$ if $e > 0$ and $(\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \notin P$.

We say that a point $\boldsymbol{p}$ (resp., a ray $\boldsymbol{r}$) *saturates* a constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b$ if and only if $\langle \boldsymbol{a}, \boldsymbol{p} \rangle = b$ (resp., $\langle \boldsymbol{a}, \boldsymbol{r} \rangle = 0$). For any point $\boldsymbol{p}$ and constraint system $\mathcal{C}$, we define

$$\mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}) \stackrel{\text{def}}{=} \{\, c \in \mathcal{C} \mid \boldsymbol{p} \text{ saturates } c \,\};$$

and, for any constraint c and generator system $\mathcal{G} = (R, P)$, we define

$$\mathrm{sat\_gen}(c, \mathcal{G}) \stackrel{\mathrm{def}}{=} \big(\{\, \boldsymbol{r} \in R \mid \boldsymbol{r} \text{ saturates } c \,\}, \{\, \boldsymbol{p} \in P \mid \boldsymbol{p} \text{ saturates } c \,\}\big).$$

Definition 9. *Let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R} \in \mathbb{CP}_{n+1}$. A constraint c is* ϵ-redundant *in $\mathcal{C}$ if $c \in \mathcal{C}_>$ and at least one of the following conditions holds:*

$$\mathrm{sat\_gen}\big(c, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq (\mathcal{G}_R, \varnothing);$$
$$\exists c' \in \mathcal{C}_> \setminus \{c\} \,.\, \mathrm{sat\_gen}\big(c, (\mathcal{G}_R, \mathcal{G}_C)\big) \sqsubseteq \mathrm{sat\_gen}(c', \mathcal{G}).$$

A generator $\boldsymbol{p}$ is ϵ-redundant *in $\mathcal{G}$ if $\boldsymbol{p} \in \mathcal{G}_P$ and*

$$\exists \boldsymbol{p}' \in \mathcal{G}_P \setminus \{\boldsymbol{p}\} \,.\, \mathrm{sat\_con}(\boldsymbol{p}, \mathcal{C}_\geq) \subseteq \mathrm{sat\_con}(\boldsymbol{p}', \mathcal{C}).$$

Note that, according to the above definition, only the strict inequality encodings and the point encodings of an ϵ-polyhedron can be identified as ϵ-redundant constraints and generators, respectively. The following result shows that such a restriction is unconsequential, because all the redundant non-strict inequality encodings and all the redundant closure point encodings will be filtered away by the usual minimization procedure.

Proposition 5. *Let $\mathcal{R}, \mathcal{R}' \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and $\mathcal{R}' \Rrightarrow_\epsilon \mathcal{P}$. Then*

1. *for any constraint $c = \big(\langle \boldsymbol{a}, \boldsymbol{x} \rangle + 0 \cdot \epsilon \geq b\big)$, $\mathcal{R} \subseteq \mathrm{con}\big(\{c\}\big)$ if and only if $\mathcal{R}' \subseteq \mathrm{con}\big(\{c\}\big)$;*
2. *for any vector $\boldsymbol{p} = (\boldsymbol{v}^{\mathrm{T}}, 0)^{\mathrm{T}} \in \mathbb{R}^{n+1}$, $\boldsymbol{p} \in \mathcal{R}$ if and only if $\boldsymbol{p} \in \mathcal{R}'$.*

The next proposition shows that ϵ-redundant constraints and generators can be safely removed from the descriptions of an ϵ-polyhedron without affecting the represented NNC polyhedron. Also, if the ϵ-polyhedron is constraint- or generator-biased, it remains constraint- or generator-biased, respectively.

Proposition 6. *Let $\Rrightarrow_Y \in \{\Rrightarrow_\epsilon, \Rrightarrow_C, \Rrightarrow_G\}$. Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_Y \mathcal{P} \neq \varnothing$ and let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal form. Then the following hold:*

1. *If c is ϵ-redundant in $\mathcal{C}$, then c is unmatched in $\mathcal{C}$ and*
$$\mathrm{con}\big(\mathcal{C} \setminus \{c\} \cup \{\epsilon \leq 1\}\big) \Rrightarrow_Y \mathcal{P}.$$
2. *If $\boldsymbol{p}$ is ϵ-redundant in $\mathcal{G} = (R, P)$, then $\boldsymbol{p}$ is unmatched in $\mathcal{G}$ and*
$$\mathrm{gen}\Big(\big(R, P \setminus \{\boldsymbol{p}\}\big)\Big) \Rrightarrow_Y \mathcal{P}.$$

If there are no ϵ-redundant constraints or generators, then the constraint or generator system, respectively, is in strong minimal form.

Proposition 7. *Let $\mathcal{R} \in \mathbb{CP}_{n+1}$ and $\mathcal{P} \in \mathbb{P}_n$ be such that $\mathcal{R} \Rrightarrow_\epsilon \mathcal{P} \neq \varnothing$ and let $(\mathcal{C}, \mathcal{G}) \equiv \mathcal{R}$ be a DD pair in minimal form. Then the following hold:*

1. *If $\mathcal{C}$ contains no ϵ-redundant constraint, then it is in* smf*;*
2. *If $\mathcal{G}$ contains no ϵ-redundant generator, then it is in* smf*.*

It must be stressed that the above generalizations of the results regarding strong minimal forms to any ϵ-polyhedron are extremely important from the point of view of efficiency. As a matter of fact, a lot of ϵ-redundant constraints and generators may be produced by a few applications of the usual operators, even when starting from descriptions that are in strong minimal form.

To exemplify such a possibility, in Table 2 we report the results obtained for a rather simple experimental evaluation, for which we have adopted the new generator-biased implementation made available by the *Parma Polyhedra Library* [1, 3]. The table has twelve rows in four groups of three. For each triple of rows, we considered four NNC polyhedra $\mathcal{P}_i$ defined by an extended generator system $\mathcal{G}_i = (\varnothing, P_i, C_i)$. All four $\mathcal{G}_i$, which are in minimal form, have the same cardinalities for the P_i and the C_i and these are given in the 1st column. The goal is to compute the NNC polyhedron $\mathcal{P} = (\mathcal{P}_1 \cap \mathcal{P}_2) \uplus (\mathcal{P}_3 \cap \mathcal{P}_4)$ and each row in the triple achieves this by following a different evaluation strategy.

$\#P_i + \#C_i$	eval	Inters ($\#\mathcal{C}_i$)		Poly-hull ($\#\mathcal{G}_{ij}$)		Final result ($\#\mathcal{C}$)			
		1st arg	2nd arg	1st arg	2nd arg	res	smf	time	time-smf
4 + 8	a	48	48	131	77	356	**56**	0.91	0.01
	b	**32**	**32**	**40**	**17**	156	**56**	0.08	0.00
	c	48	**32**	132	**17**	251	**56**	0.16	0.00
8 + 8	a	62	62	209	125	537	**59**	2.29	0.01
	b	**36**	**36**	**50**	**21**	308	**59**	0.25	0.00
	c	62	**36**	190	**21**	332	**59**	0.37	0.00
8 + 10	a	132	132	414	305	2794	**227**	118.64	0.45
	b	**68**	**68**	**58**	**25**	1084	**227**	1.42	0.06
	c	132	**68**	261	**25**	1423	**227**	3.96	0.08
16 + 10	a	178	178	697	657	5078	**235**	932.72	2.07
	b	**80**	**80**	**78**	**29**	1775	**235**	5.24	0.14
	c	178	**80**	418	**29**	1238	**235**	9.48	0.08

Table 2. Exploiting smf's to improve the efficiency of the computation.

For all evaluation strategies, in order to compute the two intersections, we first obtain the constraint systems $\mathcal{C}_i$; the 3rd and 4th columns of the table give the cardinalities of each $\mathcal{C}_i$, where the column labeled '1st arg' indicates $\#\mathcal{C}_1$ and $\#\mathcal{C}_3$ (which are always the same) while that labeled '2nd arg' indicates $\#\mathcal{C}_2$ and $\#\mathcal{C}_4$ (which are also always the same). We then compute the two intersections and obtain the generator systems $\mathcal{G}_{12}$ and $\mathcal{G}_{34}$, whose cardinalities are reported

in the 5th and 6th columns. Then, we compute the poly-hull $\mathcal{P}$. In the last four columns we report: the cardinality of the constraint system obtained for $\mathcal{P}$; the same, but after the removal of the ϵ-redundant constraints; the time spent by the overall computation; the time spent to compute the final smf. Rows labeled 'a' correspond to the usual evaluation strategy, where no smf is computed. In this case, the two intersections are computed incrementally: namely, we start from the DD pair of the first argument and incrementally add the constraints of the second argument, keeping the generator system of the result up-to-date so that it will be ready for the following poly-hull computation. Then, we compute the poly-hull, again incrementally. In the rows labeled 'b' we report the outcomes of an evaluation strategy that exploits the possibility of obtaining the smf's of both arguments before each operation (in the table, the cardinalities computed after the application of strong minimization are shown in boldface). Note that, in our current implementation, such a strategy does not fit very well with the adoption of the incremental approach, because after the computation of the smf for one description we no longer have a DD pair [3]. Therefore, after computing the smf of the first argument of each operation, the corresponding dual description has to be recomputed from scratch. The rows labeled 'c' report the outcomes of an intermediate evaluation strategy, where we only compute the smf's of the second argument of each operation, so that the incremental approach can still be adopted. For the examples considered, the latter strategy results in slightly smaller, but still impressive, performance improvements.

Even though the considered examples are not meant to provide a faithful representation of typical computation patterns, we can make a couple of observations based on these experiments. There may be many ϵ-redundant constraints/generators, and their removal can lead to dramatic speed-ups. Moreover, the identification of ϵ-redundant constraints/generators has a negligible cost (see the last column in Table 2) so that the number of ϵ-redundant elements contained in a description can be efficiently computed at run time; based on this, it is always possible to dynamically select the evaluation strategy that is likely to result in a more efficient computation. We believe that the third strategy, by preserving incrementality, is a safe and generally rewarding choice.

5 Conclusion

Convex polyhedra provide the basis for several abstractions used in static analysis and computer-aided verification of complex system. Some of these applications require the manipulation of convex polyhedra that are not necessarily closed. In a previous paper we proposed an elegant way of decoupling the essential geometric features of NNC polyhedra from their traditional implementation. This separation, besides providing a natural and easy to use interface, enables the search for new implementation techniques and makes their eventual integration into existing software libraries seamless (i.e., transparent to the client application). In this work we have shown that the standard implementation of NNC polyhedra, which happens to be biased for constraint-intensive computations,

has a dual. We have completely defined this new, previously unknown, implementation and showed that it is biased for generator-intensive computations. Moreover, we have provided a generalization of a notion of strong minimal form that is applicable to both constraint- and generator-biased implementations. We have also shown that this general notion of strong minimization can have a dramatic effect on the size of the representations and, thus, on the efficiency of the algorithms operating upon them.

The encoding based on G-ϵ-polyhedra has dual properties with respect to the one based on C-ϵ-polyhedra. In particular, using a C-ϵ-polyhedron, the encoding of an NNC polyhedron may require a similar number of constraints but as many as twice the number of generators, while, using a G-ϵ-polyhedron, it may require a similar number of generators but as many as twice the number of constraints. We have extended the *Parma Polyhedra Library* [1, 3], a modern C++ library for the manipulation of convex polyhedra, so as to implement NNC polyhedra both with the constraint- and the generator-biased encodings. This enabled us to perform some very preliminary experiments on purely synthetic benchmarks. It seems likely that the performance of one encoding with respect to the other will depend on the particular application and, more specifically, on the kind of polyhedra and operations that are more common in that application.

For future work, given the dual characteristics of the two representations, it would be interesting to investigate whether efficient techniques can be devised so as to use both constraint- and generator-biased encodings, switching dynamically from one to the other in an attempt to maximize performance.

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