

Nets, relations and linking diagrams

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Abstract. In recent work, the author and others have studied compositional algebras of Petri nets. Here we consider mathematical aspects of the pure linking algebras that underly them. We characterise composition of nets without places as the composition of spans over appropriate categories of relations, and study the underlying algebraic structures.

Introduction

Linking structures are ubiquitous in Computer Science, Logic and Mathematics. Amongst many examples, we mention Kelly-Laplaza graphs for compact closed categories [13] and proof nets [10]. Linking diagrams¹ underly string diagrams [12, 19] that are used to characterise the arrows of various kinds of free categories. Similar structures have been used by Computer Scientists to develop foundational algebras for composing software components [2, 3]. Theoretical work has led to tool support for reasoning about different kinds of string diagrams [14, 22].

In [4, 5, 20, 21] the author and others have studied compositional algebras of Petri nets. The two main variants, studied in detail in [5], are *C/E nets with boundaries* and *P/T nets with boundaries*. Nets without places are pure algebras of linkings; we show in this paper that they are, respectively, the arrows of two categories $\mathbf{Sp}(\mathbf{Rel}_f^c)$ and $\mathbf{Spr}(\mathbf{Rel}_f^M)^2$. Recently, string diagrams and closely related algebraic structures have also been used to reason about quantum computation [1, 7, 18].

Both categories are generated from a set of basic components, which are the building blocks of *two different* monoid-comonoid structures on the underlying categories. The two structures arise, roughly, from the elementary setting of cospans and spans of finite sets.

In an effort to capture several different kinds of linking algebras, Hughes [11] introduced the category \mathbf{Link} of spans over \mathbf{iRel} the category of injective relations, which has pullbacks. Pullbacks are obtained by considering paths, called *minimal synchronisations*, in the corresponding linking diagrams. Similar ideas are used here in order to construct pullbacks in \mathbf{Rel}_f^c , the category of *relations with contention* and weak pullbacks in \mathbf{Rel}_f^M , the category of multirelations. In this paper we study only finite linkings but the category of spans of relations with contention is more expressive than the category of spans of injective relations: the finite counterpart of Hughes' category \mathbf{Link} embeds into $\mathbf{Sp}(\mathbf{Rel}_f^c)$.

¹ We use this terminology loosely to mean “string diagrams without boxes.”

² The notation $\mathbf{Sp}(-)$ means “not quite the category of spans,” as the objects are the natural numbers, instead of arbitrary sets. Similarly $\mathbf{Spr}(-)$ is “not quite the category of relational spans,” where relational means that the two legs are jointly mono. Both categories are PROPs [16, 17].

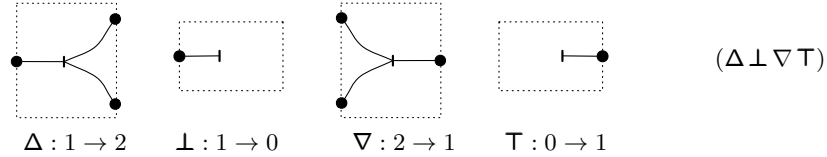
Structure of the paper. In §1 we introduce the two monoid-comonoid structures that arise from considering cospans and spans of finite sets. In §2 we introduce sets and relations with contention, and show that the category of the latter has pullbacks. This allows us, in §3 to consider the category $\mathbf{Sp}(\mathbf{Rel}_f^c)$, a universe where both the monoid-comonoid structures can be considered. In §4 we discuss multirelations and construct weak pullbacks, which we then use in §5 to consider another universe where both the monoid-comonoid structures exist and interact.

Notational conventions. Relations from X to Y are identified with functions $X \rightarrow 2^Y$. For $k \in \mathbb{N}$ we abuse notation and denote the k th finite ordinal $\{0, 1, \dots, k-1\}$ with k . For sets X, Y , $X + Y \stackrel{\text{def}}{=} \{(x, 0) \mid x \in X\} \cup \{(y, 1) \mid y \in Y\}$. Functions are labelled with $!$ when there is a unique function with that particular domain and codomain, $tw : 2 \rightarrow 2$ is the function $tw(0) = 1$ and $tw(1) = 0$. Given a function $f : X \rightarrow Y$, $[f] \subseteq X \times Y$ is its graph: $[f] \stackrel{\text{def}}{=} \{(x, fx) \mid x \in X\}$. Given a relation $R \subseteq X \times Y$, $R^{\text{op}} \subseteq Y \times X$ is the opposite relation.

1 Components of linking diagrams

Let $\mathbf{Csp}(\mathbf{Set}_f)$ be the category³ with objects the natural numbers, and arrows isomorphism classes of cospans $k \rightarrow x \leftarrow l$, where k and l are considered as finite ordinals. Composition is obtained via pushout in \mathbf{Set}_f , associativity follows from the universal property. Given $k_1 \rightarrow m_1 \leftarrow l_1$ and $k_2 \rightarrow m_2 \leftarrow l_2$, the tensor product is $k_1 + k_2 \rightarrow m_1 + m_2 \leftarrow l_1 + l_2$.

The following diagrams represent certain arrows in $\mathbf{Csp}(\mathbf{Set}_f)$. They



have representatives $1 \xrightarrow{\text{id}} 1 \xleftarrow{!} 2$, $1 \xrightarrow{\text{id}} 1 \xleftarrow{!} 0$, $2 \xrightarrow{!} 1 \xleftarrow{\text{id}} 1$ and $0 \xrightarrow{!} 1 \xleftarrow{\text{id}} 1$.

Our graphical notation calls for further explanation: within the diagrams, each *link*—an undirected multiedge—represents an element of the carrier set, its connections to *boundary ports* (elements of the ordinals on the boundary) are determined in $\mathbf{Csp}(\mathbf{Set}_f)$ by the functions from the ordinals that represent the boundaries. Each link has a small perpendicular mark; this is used to distinguish between different links within diagrams.

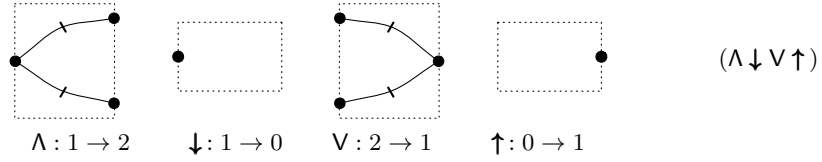
The definition of $\mathbf{Csp}(\mathbf{Set}_f)$ enforces some structural restrictions on links. Indeed, each boundary port must be connected to exactly one link; ie no two links can be connected to the same boundary port. Any link, however, can be connected to several ports on each boundary.

Now consider $\mathbf{Sp}(\mathbf{Set}_f)$, the category with objects the natural numbers, and arrows isomorphism classes of spans $k \leftarrow x \rightarrow l$, where k and l are considered as finite ordinals. Composition is obtained via pullback in

³ Not quite the category of cospans. Again, this is a PROP.

\mathbf{Set}_f , and associativity is again guaranteed by a universal property, this time of pullbacks. Again, $+$ gives a tensor product.

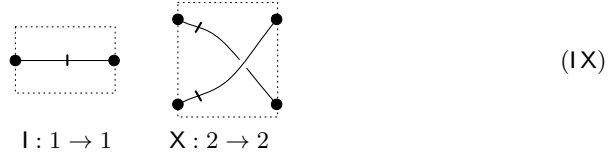
The following diagrams represent certain arrows in $\mathbf{Sp}(\mathbf{Set}_f)$. They



have representatives $1 \xleftarrow{!} 2 \xrightarrow{\text{id}} 2$, $1 \xleftarrow{!} 0 \xrightarrow{\text{id}} 0$, $2 \xleftarrow{\text{id}} 2 \xrightarrow{!} 1$ and $0 \xleftarrow{\text{id}} 0 \xrightarrow{!} 1$.

In the diagrams, the links again represent elements of the carrier set but connections to boundary ports are now given by the functions *from* the carrier *to* the boundaries. Due to the definition of $\mathbf{Sp}(\mathbf{Set}_f)$, there are again structural restrictions: each link is connected to exactly one port on each boundary. Any port, however, can be connected to many links.

The following diagrams represent certain arrows in $\mathbf{Csp}(\mathbf{Set}_f)$ and $\mathbf{Sp}(\mathbf{Set}_f)$. As (isomorphism classes of) cospans they are $1 \rightarrow 1 \leftarrow 1$,



$2 \xrightarrow{tw} 2 \leftarrow 2$, as spans they are $1 \leftarrow 1 \rightarrow 1$, $2 \leftarrow 2 \xrightarrow{tw} 2$.

1.1 The algebra of $\mathbf{Csp}(\mathbf{Set}_f)$

In Fig. 1 we give some of the equations satisfied by the algebra generated from the components $(\Delta \perp \nabla \top)$ and (IX) in $\mathbf{Csp}(\mathbf{Set}_f)$: (ΔUC) and (ΔA) show that Δ is the comultiplication of a cocommutative comonoid. The symmetric equations hold for ∇ , meaning that it is part of a commutative monoid structure. The Frobenius axioms (F) [6, 15] hold, and the algebra is separable (S). In fact $\mathbf{Csp}(\mathbf{Set}_f)$ is the free PROP on $(\Delta \perp \nabla \top)$ satisfying such axioms, where (F), (S) can be understood as witnessing a distributive law of PROPs; see [16] for the details. In (CC) we indicate how the (self dual) compact closed structure of $\mathbf{Csp}(\mathbf{Set}_f)$ arises.

1.2 The algebra of $\mathbf{Sp}(\mathbf{Set}_f)$

In Fig. 2 we exhibit some equations satisfied by the components $(\Lambda \downarrow \vee \uparrow)$ and (IX) in $\mathbf{Sp}(\mathbf{Set}_f)$: (ΛUC) and (ΛA) show that Λ is the multiplication of a cocommutative comonoid, similarly the symmetric equations, which we do not illustrate, show that that \vee is a commutative monoid. Differently from Fig. 1, here the Frobenius equations do not hold; but rather the equations of commutative and cocommutative bialgebras: in (B), $(\vee \downarrow)$ and $(\Lambda \vee)$ we show how the monoid and comonoid structures interact in

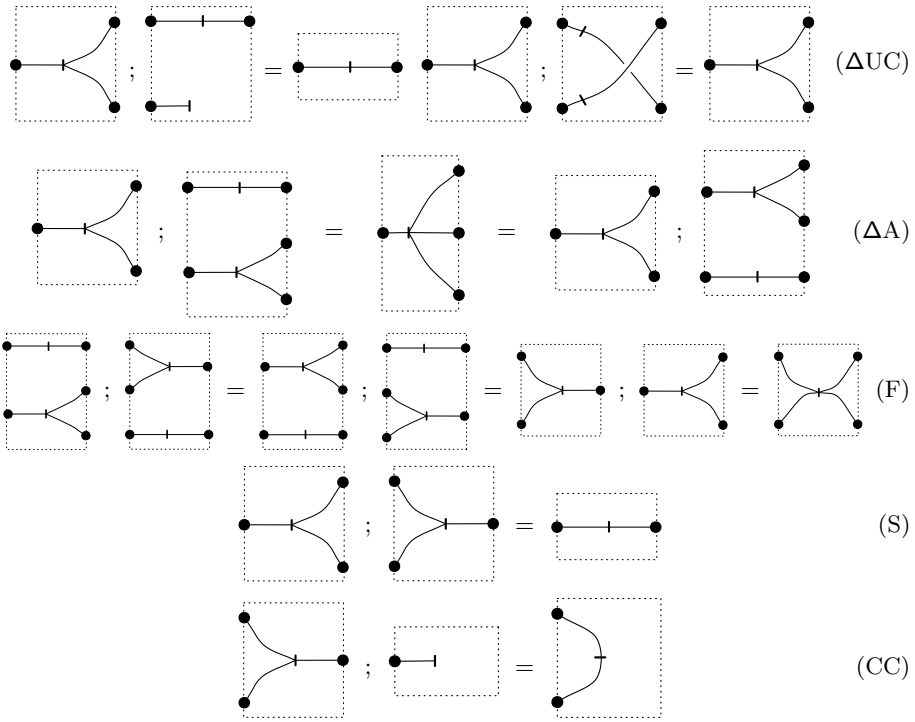


Fig. 1. Equations in $\text{Csp}(\text{Set}_f)$.

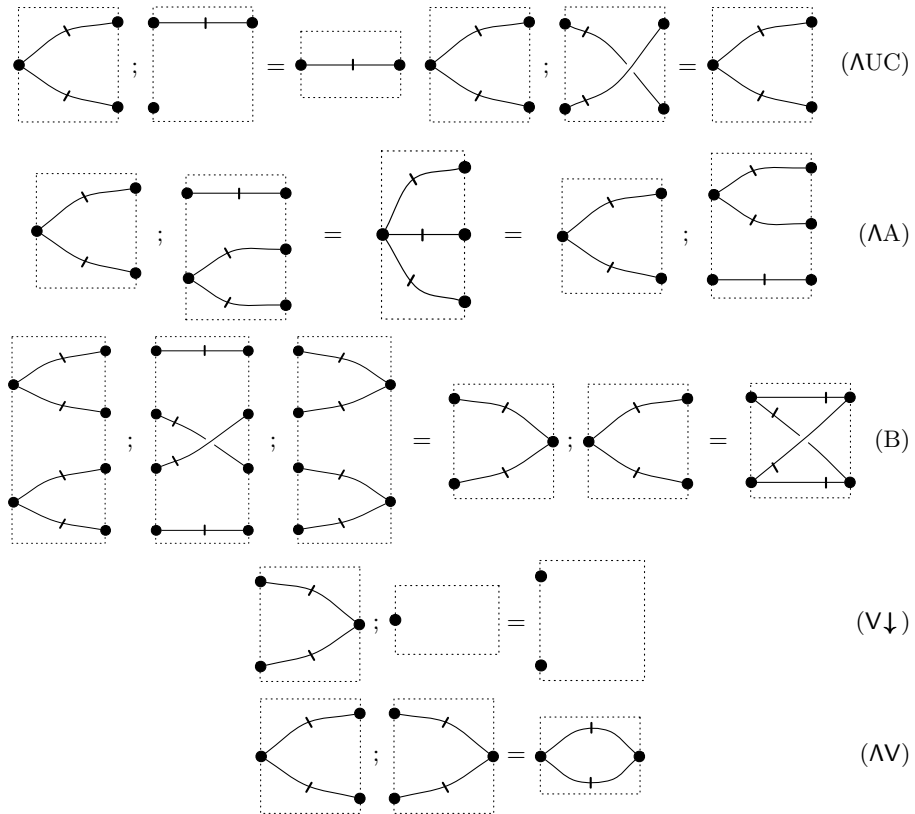


Fig. 2. Equations in $\text{Sp}(\text{Set}_f)$.

$\mathbf{Sp}(\mathbf{Set}_f)$. In fact, $\mathbf{Sp}(\mathbf{Set}_f)$ is the free PROP on $(\wedge \downarrow \vee \uparrow)$ satisfying the equations of commutative and cocommutative bialgebras, and the bialgebra axiom can be understood as a distributive law of PROPs, see [16].

1.3 Bringing it all together

Note that none of the diagrams in $(\Delta \perp \nabla \top)$ represent valid spans: for instance the link in Δ connects to two different ports on its right boundary, and the link in \perp does not connect to any port on its right boundary. Similarly, none of $(\wedge \downarrow \vee \uparrow)$ represent valid cospans. Thus, for mundane “expressivity” reasons, $(\Delta \perp \nabla \top)$ are not arrows of $\mathbf{Sp}(\mathbf{Set}_f)$, and vice-versa, $(\wedge \downarrow \vee \uparrow)$ are not arrows of $\mathbf{Csp}(\mathbf{Set}_f)$. The remit of this paper is to study how these two commutative monoid-comonoid structures interact together in universes that are expressive enough to accommodate them.

For example, instead of studying cospans and spans of *functions*, one could consider spans (or cospans) of *relations*. Indeed, it is not difficult to check that all of the components $(\Delta \perp \nabla \top)$, $(\wedge \downarrow \vee \uparrow)$ and (IX) are spans of relations of finite sets. The problem, of course, is that \mathbf{Rel}_f , the category of finite sets and relations, does not have pullbacks nor pushouts: it is thus not clear how to define the composition of such linking diagrams.

In the following sections we study two different universes that are expressive enough to contain $(\Delta \perp \nabla \top)$, $(\wedge \downarrow \vee \uparrow)$ and (IX) and the intriguing, different ways in which the two monoid/comonoid structures interact in the universes. They arose through the study of compositional algebras of Petri nets with boundaries [4, 5, 20, 21].

2 Sets with contention

In this section we introduce *sets with contention*, over which one can define a category of relations that has pullbacks, and is expressive enough to accommodate the components $(\Delta \perp \nabla \top)$, $(\wedge \downarrow \vee \uparrow)$ and (IX) .

A *set with contention*, or *c-set*, is a pair (X, \bowtie_X) , where X is a set and $\bowtie_X \subseteq X \times X$ is a reflexive ($\forall x \in X. (x \bowtie_X x)$) and symmetric ($\forall x, y \in X. (x \bowtie_X y) \Rightarrow (y \bowtie_X x)$) relation called *contention*.⁴ The complement relation \parallel_X is called *independence*. To describe a *c-set* it is thus of course enough to specify either contention or independence. Sets with contention of the form (X, δ_X) , where $\delta_X = \{(x, x) \mid x \in X\}$, are said to be *discrete*. We will normally write simply X for the pair (X, \bowtie_X) .

A *morphism of c-sets* $f : X \rightarrow Y$ is a function $f : X \rightarrow Y$ such that:

$$\forall x, x' \in X. f(x) \bowtie_Y f(x') \Rightarrow x \bowtie_X x' \quad (1)$$

(or equivalently $\forall x, x' \in X. x \parallel_X x'$ implies $f(x) \parallel_Y f(x')$.) The category of finite *c-sets* and their morphisms is denoted \mathbf{Set}_f^c .

Given *c-sets* X_0 and X_1 , $X_0 + X_1$ is the *c-set* with $X_0 + X_1$ as its underlying set and $(x, i) \bowtie_{X_0+X_1} (y, j)$ iff $i = j$ and $x \bowtie_{X_i} y$. This is the categorical coproduct in \mathbf{Set}_f^c .

⁴ A useful intuition is that links carry signals. When two links are in contention they cannot transmit concurrently. With this intuition $(\Delta \perp \nabla \top)$ are copy and forget operations, while $(\wedge \downarrow \vee \uparrow)$ are non-deterministic switches and “failure.”

Given a c -set X , $U \subseteq X$ is said to be *independent* when

$$\forall u, u' \in U. u \bowtie_X u' \Rightarrow u = u'. \quad (2)$$

Let $\mathcal{P}_c X$ denote the set of independent subsets of X . There is functor $\mathcal{P}_c : \mathbf{Set}_f^c \rightarrow \mathbf{Set}_f^c$ that takes a c -set X to the set of independent subsets $\mathcal{P}_c X$, with contention between subsets defined:

$$U \bowtie_{\mathcal{P}_c X} V \text{ iff } \exists u \in U, v \in V, u \bowtie_X v.$$

Note that independent subsets are closed under intersection and set difference: indeed, if $U' \subseteq U$ and U is independent then also U' is independent. They are not, in general, closed under union.

If $f : X \rightarrow Y$ is a morphism, then letting

$$\mathcal{P}_c f(U) \stackrel{\text{def}}{=} \{ fu \mid u \in U \}$$

defines a morphism $\mathcal{P}_c f : \mathcal{P}_c X \rightarrow \mathcal{P}_c Y$ in \mathbf{Set}_f^c , since:

- (i) given U , for all $u, u' \in U$ if $f(u) \bowtie_Y f(u')$ means that $u \bowtie_X u'$. But U is independent, and thus $u = u'$ and $f(u) = f(u')$, thus $\mathcal{P}_c f(U)$ is an independent subset of Y (recall (2)).
- (ii) if $\mathcal{P}_c f(U) \bowtie_Y \mathcal{P}_c f(V)$ then there exists $u \in U, v \in V$, such that $f(u) \bowtie_Y f(v)$, so $u \bowtie_X v$ and thus $U \bowtie_X V$, thus $\mathcal{P}_c f$ satisfies (1).

2.1 Relations with contention

There are morphisms $\mu_X : \mathcal{P}_c^2 X \rightarrow \mathcal{P}_c X$ with $\{U_i\} \mapsto \bigcup_i U_i$ and a morphism $\eta_X : X \rightarrow \mathcal{P}_c X$. It is not difficult to check that they are natural transformations that satisfy the monad axioms.

Let $\mathbf{Rel}_f^c \stackrel{\text{def}}{=} \mathbf{Kl}(\mathcal{P}_c)$ of relations with contention, or c -relations, be the Kleisli category with objects finite c -sets. Arrows from X to Y are morphisms $f : X \rightarrow \mathcal{P}_c Y$ in \mathbf{Set}_f^c , which we will sometimes denote $f : X \rightrightarrows Y$. Given a morphism $f : X \rightarrow \mathcal{P}_c Y$ in \mathbf{Set}_f^c (or equivalently, a morphism of \mathbf{Rel}_f^c), $f^\# : \mathcal{P}_c X \rightarrow \mathcal{P}_c Y$ is the morphism $f^\# U \stackrel{\text{def}}{=} \bigcup_{u \in U} fu$.

The following lemma is useful when calculating in \mathbf{Rel}_f^c . It does not hold in \mathbf{Rel}_f , the category of ordinary finite sets and relations.

Lemma 21 *Suppose $f : X \rightarrow \mathcal{P}_c Y$ in \mathbf{Set}_f^c . Then, given $U, U' \in \mathcal{P}_c X$ with $U \subseteq U'$, $f^\#(U' \setminus U) = f^\#(U') \setminus f^\#(U)$. Also, given $U, V, V' \in \mathcal{P}_c X$, with $V \subseteq U, V' \subseteq U$, we have $f^\#(V \cap V') = f^\#(V) \cap f^\#(V')$.*

Proof. Since U' is independent, $\{fu\}_{u \in U'}$ is a family of disjoint, independent subsets of Y . Similarly $V \cup V'$ is independent, since they are both subsets of an independent set; and $\{fu\}_{u \in V \cup V'}$ is a family of disjoint, independent subsets of Y . Disjointness implies the desired conclusions. \square

2.2 Pullbacks in \mathbf{Rel}_f^c

Suppose that $f : A \rightrightarrows X$ and $g : B \rightrightarrows X$ in \mathbf{Rel}_f^c . Given $U \in \mathcal{P}_c A, V \in \mathcal{P}_c B$, say that (U, V) is a (f, g) -*synchronisation*⁵ if $f^\#U = g^\#V$.

⁵ Hughes [11] uses the term synchronisation in a similar context, and the term has been used in [4, 5, 20] to compose Petri nets with boundaries.

We will typically infer f and g from the context and write ‘ $\langle U \Downarrow V \rangle$ ’ as shorthand for ‘a synchronisation (U, V) ’. Synchronisations inherit an ordering from the subset ordering, pointwise:

$$\langle U \Downarrow V \rangle \subseteq \langle U' \Downarrow V' \rangle \stackrel{\text{def}}{=} U \subseteq U' \wedge V \subseteq V'.$$

The *trivial* synchronisation is $\langle \emptyset \Downarrow \emptyset \rangle$. A synchronisation $\langle U \Downarrow V \rangle$ is said to be *minimal* when it is not trivial and for all $\langle U' \Downarrow V' \rangle$ such that $\langle U' \Downarrow V' \rangle \subseteq \langle U \Downarrow V \rangle$, either $\langle U' \Downarrow V' \rangle$ is trivial or equal to $\langle U \Downarrow V \rangle$.

Let $\text{minsnc}(f, g)$ be the set of minimal synchronisations of f and g . We can define contention on this set by letting

$$\langle U \Downarrow V \rangle \bowtie_{\text{minsnc}(f, g)} \langle U' \Downarrow V' \rangle \stackrel{\text{def}}{=} U \bowtie_{\mathcal{P}A} U' \vee V \bowtie_{\mathcal{P}B} V'.$$

It follows that have the following commutative diagram in \mathbf{Rel}_f^c

$$\begin{array}{ccc} & \text{minsnc}(f, g) & \\ p \swarrow & & \searrow q \\ A & & B \\ f \searrow & & \swarrow g \\ & X & \end{array} \quad (3)$$

where $p\langle U \Downarrow V \rangle = U$ and $q\langle U \Downarrow V \rangle = V$. The following observations will lead us to conclude in Lemma 24 that the diagram is a pullback in \mathbf{Rel}_f^c .

Synchronisations are not in general closed under (pointwise) union, because if $\langle U \Downarrow V \rangle$ and $\langle U' \Downarrow V' \rangle$ then in general it is not true that $U \cup U' \in \mathcal{P}_c A$ and $V \cup V' \in \mathcal{P}_c B$. It is true, however, that the union of any set of minimal synchronisations contained in any synchronisation is again a synchronisation: this is guaranteed by the following.

Lemma 22 *Suppose that $\langle U' \Downarrow V' \rangle \neq \langle U'' \Downarrow V'' \rangle$ are minimal synchronisations contained in $\langle U \Downarrow V \rangle$. Then $U' \cap U'' = \emptyset$ and $V' \cap V'' = \emptyset$.*

Proof. By the conclusion of Lemma 21, $f^\#(U \cap U') = f^\#U \cap f^\#U' = g^\#V \cap g^\#V' = g^\#(V \cap V')$, so $\langle U' \cap U'' \Downarrow V' \cap V'' \rangle$; by minimality of $\langle U' \Downarrow V' \rangle$ and $\langle U'' \Downarrow V'' \rangle$ it follows that $\langle U' \cap U'' \Downarrow V' \cap V'' \rangle$ is trivial. \square

Lemma 23 *$\langle U \Downarrow V \rangle$ is the union of min. synchronisations it contains.*

Proof. Let $\{\langle U_i \Downarrow V_i \rangle\}_{i \in I}$ be the set of minimal synchronisations contained in $\langle U \Downarrow V \rangle$ and $\langle U' \Downarrow V' \rangle \stackrel{\text{def}}{=} \bigcup_i \{\langle U_i \Downarrow V_i \rangle\}$, then clearly we have $\langle U' \Downarrow V' \rangle \subseteq \langle U \Downarrow V \rangle$. Let $U'' = U \setminus U'$ and $V'' = V \setminus V'$. Now, using the conclusion of Lemma 21, $\langle U'' \Downarrow V'' \rangle$, and thus it is either null or it contains a minimal synchronisation. But $\{\langle U_i \Downarrow V_i \rangle\}_{i \in I}$ contains all minimal synchronisations in $\langle U \Downarrow V \rangle$; thus $U'' = V'' = \emptyset$ and we are finished. \square

Lemma 24 *The square (3) is a pullback diagram in \mathbf{Rel}_f^c .*

Proof. Suppose Z is a c -set and $\alpha : Z \rightarrow A$, $\beta : Z \rightarrow B$ are morphisms in \mathbf{Rel}_f^c such that $f\alpha = g\beta$. In particular, this means that for all $z \in Z$, we have $\langle \alpha z \Downarrow \beta z \rangle$. Define $h : Z \rightarrow \text{minsnc}(f, g)$ by letting hz be the family of minimal synchronisations contained in $\langle \alpha z \Downarrow \beta z \rangle$. This is a independent

set, due to Lemma 22, and the fact that αz and βz are independent. Then, by the conclusion of Lemma 23, $ph = \alpha$ and $qh = \beta$.

If another h' satisfies $ph' = \alpha$ and $qh' = \beta$ then there exists a family of minimal synchronisations $h'z = \{\langle U_i \vee V_i \rangle\}_{i \in I}$ such that $\bigcup_i U_i = \alpha z$ and $\bigcup_i V_i = \beta z$. By the conclusion of Lemma 22 this family must be hz . \square

3 The algebra of $\mathbf{Sp}(\mathbf{Rel}_f^c)$

In this section we consider a category with enough structure for all of $(\Delta \perp \nabla \top)$, $(\wedge \downarrow \vee \uparrow)$ and (\mathbf{IX}) . It has been considered as part of a compositional algebra of C/E (1 bounded) nets [5,20]—indeed, it is the category of C/E nets with boundaries, without net places, up to isomorphism.

Consider the category $\mathbf{Sp}(\mathbf{Rel}_f^c)$, that has objects the natural numbers and arrows $k \rightarrow l$ isomorphism classes of spans $k \xleftarrow{f} (X, \bowtie_X) \xrightarrow{g} l$ in \mathbf{Rel}_f^c , where k and l are considered as discrete c -sets. Composition is via pullback in \mathbf{Rel}_f^c ; associativity follows from the universal property. There is a tensor product, given by $+$.

$\mathbf{Sp}(\mathbf{Rel}_f^c)$ has enough structure for $(\Delta \perp \nabla \top)$, $(\wedge \downarrow \vee \uparrow)$ and (\mathbf{IX}) . Indeed, $(\Delta \perp \nabla \top)$ are, respectively, spans $1 \xleftarrow{[\text{id}]} 1 \xrightarrow{[!]^{\text{op}}} 2$, $1 \xleftarrow{[\text{id}]} 1 \xrightarrow{[!]^{\text{op}}} 0$, $2 \xleftarrow{[!]^{\text{op}}} 1 \xrightarrow{[\text{id}]} 1$ and $0 \xleftarrow{[!]^{\text{op}}} 1 \xrightarrow{[\text{id}]} 1$. Similarly, (\mathbf{IX}) are spans $1 \xleftarrow{[\text{id}]} 1 \xrightarrow{[\text{id}]} 2$ and $2 \xleftarrow{[\text{id}]} 2 \xrightarrow{[tu]} 2$. Indeed, $\mathbf{Csp}(\mathbf{Set}_f)$ embeds into $\mathbf{Sp}(\mathbf{Rel}_f^c)$.

Theorem 1. *There is a faithful functor $E : \mathbf{Csp}(\mathbf{Set}_f) \rightarrow \mathbf{Sp}(\mathbf{Rel}_f^c)$ that is identity-on-objects.*

Proof. A cospan $k \xrightarrow{f} x \xleftarrow{g} l$ is taken to $k \xleftarrow{[f]^{\text{op}}} x \xrightarrow{[g]^{\text{op}}} l$, where k , x and l are discrete c -sets, and $[f]^{\text{op}}$, $[g]^{\text{op}}$ are the opposites of graphs of, respectively, f and g . As arrows in $\mathbf{Kl}(\mathcal{P}_c)$, $[f]^{\text{op}}u = f^{-1}u$ and $[g]^{\text{op}}u = g^{-1}u$ for any $u \in x$. Identities are clearly preserved.

We must show that composition is preserved; it suffices to show that, given $g_0 : l \rightarrow x_0$ and $f_1 : l \rightarrow x_1$, a pushout diagram of g_0, f_1 in \mathbf{Set}_f is taken to a pullback diagram in \mathbf{Rel}_f^c , as illustrated below.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & l & \\
 g_0 \swarrow & & \searrow f_1 \\
 x_0 & & x_1 \\
 r \searrow & \widehat{\quad} & \swarrow s \\
 & M &
 \end{array}
 & \longmapsto &
 \begin{array}{ccc}
 & M & \\
 [r]^{\text{op}} \swarrow & & \searrow [s]^{\text{op}} \\
 x_0 & & x_1 \\
 [g_0]^{\text{op}} \searrow & & \swarrow [f_1]^{\text{op}} \\
 & l &
 \end{array}
 \end{array} \tag{4}$$

If $M = 0$ then also $x_0 = x_1 = l = 0$ and all arrows are id_0 . Otherwise, by an inductive argument it suffices to consider the case $M = 1$. In that case, if $x_0 = 0$ then $x_1 = 1$ and $l = 0$. Then $\text{minsnc}([g_0]^{\text{op}}, [f_1]^{\text{op}}) = \{\langle \emptyset \vee 1 \rangle\}$ and we are done. The case $x_1 = 0$ is symmetric. If both $x_0, x_1 \neq 0$ then clearly $\langle x_0 \vee x_1 \rangle$. In fact, it is the only non-trivial synchronisation (and thus minimal). To see this, notice that g_0 and f_1 are surjective and therefore, if $\langle U_1 \vee V_1 \rangle$ and $\langle U_2 \vee V_2 \rangle$ are two different non-trivial synchronisations then $l_1 \stackrel{\text{def}}{=} g_0^{-1}U_1 = f_1^{-1}V_1 \neq \emptyset$ and $l_2 \stackrel{\text{def}}{=} g_0^{-1}U_2 = f_1^{-1}V_2 \neq \emptyset$, but $l_1 \cap l_2 = \emptyset$. This means that $l = l_1 + l_2 + l_3$, for some l_3 ,

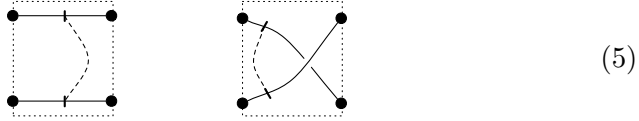
and the whole left hand side of (4) decomposes into a sum, contradicting the assumption that $M = 1$.

The inductive argument relies on sums being compatible with pull-backs in \mathbf{Rel}_f^c . This follows from the construction: minimal synchronisations of $x_0 + x'_0 \xrightarrow{[g_0+g'_0]^{op}} l + l' \xleftarrow{[f_0+f'_1]^{op}} x_1 + x'_1$ arise either as a minimal synchronisations of $[g_0]^{op}$ and $[f_1]^{op}$, or those of $[g'_0]^{op}$ and $[f'_1]^{op}$. \square

As a consequence, the equations for $(\Delta \perp \nabla \top)$ —presented in (ΔUC) , (ΔA) , (F), (S) and (CC)— also hold in $\mathbf{Sp}(\mathbf{Rel}_f^c)$.

Also $(\Lambda \downarrow \vee \uparrow)$ are spans of c -relations: $1 \xleftarrow{[!]} (2, 2 \times 2) \xrightarrow{[id]} 2$, $1 \xleftarrow{[!]} 0 \xrightarrow{[id]} 0$, $2 \xleftarrow{[id]} (2, 2 \times 2) \xrightarrow{[!]} 1$ and $0 \xleftarrow{[id]} 0 \xrightarrow{[!]} 1$; notice that contention is used to “encode” $(\Lambda \downarrow \vee \uparrow)$. This is necessary because the two elements of 2 must be in contention in order for $! : 2 \rightarrow 1$ to be a c -morphism.

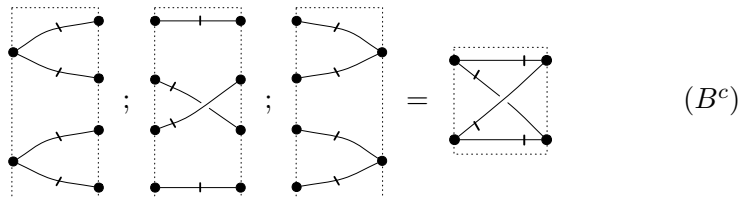
Remark 1. When considering, for instance Λ of $(\Lambda \downarrow \vee \uparrow)$ we are in a situation where two links connect to the same point on the boundary. Since any element is in contention with itself, this means that the two links must be in contention. Thus, in this example, contention between the two links is implied and we will not alter our graphical notation. We will, however, need a way to represent contention graphically when it is not implied “structurally,” and we will do this by connecting the links with dotted lines. For instance, the two diagrams below represent the spans $2 \xleftarrow{[id]} (2, 2 \times 2) \xrightarrow{[id]} 2$ and $2 \xleftarrow{[id]} (2, 2 \times 2) \xrightarrow{[tw]} 2$.



Remark 2. There is also an “embedding” $F : \mathbf{Sp}(\mathbf{Set}_f) \rightarrow \mathbf{Sp}(\mathbf{Rel}_f^c)$. A span $k \xleftarrow{f} x \xrightarrow{g} l$ is sent to the span $k \xleftarrow{[f]} (x, x \times x) \xrightarrow{[g]} l$, with the carrier set having all elements in contention. It is not difficult to check that composition is preserved, but the mapping fails to be a functor because identities are not preserved. For instance, the identity on 2 is mapped to the left diagram of (5), which is not the identity on 2 in $\mathbf{Sp}(\mathbf{Rel}_f^c)$.

The finite fragment of Hughes’ category \mathbf{Link} of spans of *injective* relations [11] lies between $\mathbf{Csp}(\mathbf{Set}_f)$ and $\mathbf{Sp}(\mathbf{Rel}_f^c)$. Indeed, spans of injective relations are expressive enough to consider all the structure of $(\Delta \perp \nabla \top)$, (IX) and the units \downarrow, \uparrow of $(\Lambda \downarrow \vee \uparrow)$; but not the comultiplication and multiplication Λ, \vee — these are not injective relations. \mathbf{Link} embeds into $\mathbf{Sp}(\mathbf{Rel}_f^c)$, thus all the equations that hold in the former hold also in the latter. We omit the details here.

Equations (ΛUC) , (ΛA) , $(\vee \downarrow)$ and $(\Lambda \vee)$ hold in in $\mathbf{Sp}(\mathbf{Rel}_f^c)$. Equation (B) does not hold: while we have



we have

$$\begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} \quad (\forall \Lambda^c)$$

and the right-hand sides are not equal as arrows of $\text{Sp}(\mathbf{Rel}_f^c)$.

In $(\Delta \downarrow \uparrow^c)$, $(\Lambda \top \perp^c)$, $(\Delta \vee^c)$ and $(\Delta \wedge^c)$ below we show how $(\Delta \perp \nabla \top)$ and $(\Lambda \downarrow \vee \uparrow)$ interact together in $\text{Sp}(\mathbf{Rel}_f^c)$. We comment on two of the more interesting equations that the interactions suggest: the right hand side of $(\Delta \vee^c)$ implies $\vee ; \Delta = (\Delta \otimes \Delta) ; (I \otimes X \otimes I) ; (\vee \otimes \vee)$, an ‘‘asymmetric’’ commutative/cocommutative bialgebra structure. The left hand side of $(\Delta \wedge^c)$ implies $\Delta ; (\wedge \otimes I) = \wedge ; (\Delta \otimes \Delta) ; (I \otimes X \otimes I) ; (I \otimes I \otimes \vee)$.

$$\begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} \quad (\Delta \downarrow \uparrow^c)$$

$$\begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} \quad (\Lambda \top \perp^c)$$

$$\begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \swarrow \\ \bullet \end{array} \quad (\Delta \vee^c)$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} ; \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad (\Delta \wedge^c)$$

All linking diagrams in $\text{Sp}(\mathbf{Rel}_f^c)$ can be obtained from the basic set of components $(\Delta \perp \nabla \top)$, $(\Lambda \downarrow \vee \uparrow)$ and (IX) , combined using the operations of composition and tensor.

Theorem 2. *Every arrow in $\text{Sp}(\mathbf{Rel}_f^c)$ decomposes into an expression consisting only of $\Delta, \perp, \nabla, \top, \wedge, \downarrow, \vee, \uparrow, I, X$, composed with $;$ and \otimes .*

Proof. Omitted.

4 Multisets and multirelations

We have seen that $\text{Sp}(\mathbf{Rel}_f^c)$ is a setting in which one can study the algebra of $(\Delta \perp \nabla \top)$, $(\Lambda \downarrow \vee \uparrow)$ and (IX) . Here we develop a second, different setting, that arises from a compositional algebra of P/T nets [5].

Given a set X , let \mathcal{M}_X denote the set of finite maps $\mathcal{U} : X \rightarrow \mathbb{N}$, ie where $\text{dom}(\mathcal{U})$ is a finite set. We call elements of \mathcal{M}_X *multisets*. We will sometimes abuse set notation to when talking about multisets; any ordinary set $U \subseteq X$ can be considered as a multiset in the obvious way:

$$\mathcal{U}x = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Given $\mathcal{U}, \mathcal{V} \in \mathcal{M}_X$, $\mathcal{U} + \mathcal{V}$ is the multiset $(\mathcal{U} + \mathcal{V})(x) = \mathcal{U}x + \mathcal{V}x$. We say $\mathcal{U} \geq \mathcal{V}$ if $\forall x. \mathcal{U}x - \mathcal{V}x \in \mathbb{N}$. If $\mathcal{U} \geq \mathcal{V}$, let $(\mathcal{U} - \mathcal{V}) \in \mathcal{M}_X$ be defined $(\mathcal{U} - \mathcal{V})x \stackrel{\text{def}}{=} \mathcal{U}x - \mathcal{V}x$. Given $k \in \mathbb{N}$ and $\mathcal{U} \in \mathcal{M}_X$, $k\mathcal{U}(x) \stackrel{\text{def}}{=} k \cdot \mathcal{U}(x)$.

\mathcal{M}_X is the action on objects of the functor $\mathcal{M}_- : \mathbf{Set} \rightarrow \mathbf{Set}$. On functions, $\mathcal{M}_f : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ is defined $\mathcal{M}_f \mathcal{U}(y) = \sum_{x \in X: f(x)=y} \mathcal{U}x$; note that since \mathcal{U} is nonzero on a finite subset of X , this is well-defined. There is a natural transformation $\mu_X : \mathcal{M}_{\mathcal{M}_X} \rightarrow \mathcal{M}_X$ that takes $\mu_X \mathcal{V}(x) = \sum_{\mathcal{V}(\mathcal{U}) \geq 0} \mathcal{V} \mathcal{U} \cdot \mathcal{U}x$ and $\eta_X : X \rightarrow \mathcal{M}_X$ where $\eta_X x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$ It is not difficult to check that $(\mathcal{M}_-, \mu, \eta)$ is a monad, commonly referred to as the multiset monad. Given $f : A \rightarrow \mathcal{M}_B$, the definition of $f^\# : \mathcal{M}_A \rightarrow \mathcal{M}_B$ follows from a simple calculation: $f^\#(\mathcal{U}) = \sum_{a \in A} (\mathcal{U}a) f(a)$.

Let $\mathbf{Rel}^{\mathcal{M}} \stackrel{\text{def}}{=} Kl(\mathcal{M}_-)$ and $\mathbf{Rel}_f^{\mathcal{M}}$ be the full subcategory of $\mathbf{Rel}^{\mathcal{M}}$ with objects the finite sets. The arrows of $\mathbf{Rel}_f^{\mathcal{M}}$ are thus functions $f : X \rightarrow \mathcal{M}_Y$ in \mathbf{Set}_f , we will sometimes write $f : X \mapsto Y$.

4.1 Multi synchronisations

Suppose that $f : A \mapsto X$ and $g : B \mapsto X$ in $\mathbf{Rel}_f^{\mathcal{M}}$. A (multi f, g) synchronisation is a pair $\langle \mathcal{U}, \mathcal{V} \rangle$ with $\mathcal{U} \in \mathcal{M}_A$ and $\mathcal{V} \in \mathcal{M}_B$ such that $f^\# \mathcal{U} = g^\# \mathcal{V}$. A synchronisation thus consists of a multiset of A together with a multiset of B that both map to the same multiset of X via $f^\#$ and $g^\#$, respectively; this notion is the multiset equivalent of the notion of synchronisation that we have considered in §2.2. We will again write $\langle \mathcal{U} \Downarrow \mathcal{V} \rangle$ as shorthand and write $\text{snc}(f, g)$ for the set of synchronisations.

Synchronisations inherit an ordering from multisets, pointwise. If we have $\langle \mathcal{U}' \Downarrow \mathcal{V}' \rangle \leq \langle \mathcal{U} \Downarrow \mathcal{V} \rangle$ then $\langle \mathcal{U} - \mathcal{U}' \Downarrow \mathcal{V} - \mathcal{V}' \rangle$: indeed $f^\#(\mathcal{U} - \mathcal{U}') = f^\# \mathcal{U} - f^\# \mathcal{U}' = g^\# \mathcal{V} - g^\# \mathcal{V}' = g^\#(\mathcal{V} - \mathcal{V}')$. Synchronisations are closed under linear combinations: if $\{\langle \mathcal{U}_i \Downarrow \mathcal{V}_i \rangle\}_{i \in I}$ and $k_i \in \mathbb{N}$ then define $\sum_i k_i \langle \mathcal{U}_i \Downarrow \mathcal{V}_i \rangle \stackrel{\text{def}}{=} \langle \sum_i k_i \mathcal{U}_i, \sum_i k_i \mathcal{V}_i \rangle$, which is clearly a synchronisation.

A set \mathbf{X} of synchronisations is *mutually incomparable* when

$$\begin{aligned} \forall \langle \mathcal{U} \Downarrow \mathcal{V} \rangle, \langle \mathcal{U}' \Downarrow \mathcal{V}' \rangle \in \mathbf{X}. \langle \mathcal{U} \Downarrow \mathcal{V} \rangle \leq \langle \mathcal{U}' \Downarrow \mathcal{V}' \rangle \vee \langle \mathcal{U}' \Downarrow \mathcal{V}' \rangle \leq \langle \mathcal{U} \Downarrow \mathcal{V} \rangle \\ \Rightarrow \langle \mathcal{U} \Downarrow \mathcal{V} \rangle = \langle \mathcal{U}' \Downarrow \mathcal{V}' \rangle. \end{aligned}$$

We need to recall a version of Dickson's lemma [8], stated in terms of synchronisations. It can be proved by a straightforward induction.

Lemma 41 (Dickson) *Suppose $f : A \mapsto X$ and $g : B \mapsto X$ in $\mathbf{Rel}_f^{\mathcal{M}}$. Any set X of mutually incomparable multi- f, g synchronisations is finite.*

Let $\text{minsnc}(f, g)$ be the set of minimal synchronisations. Clearly any two minimal synchronisations are incomparable, thus, by the conclusion of Lemma 41, $\text{minsnc}(f, g)$ is finite. In particular (6) is a commutative diagram in $\mathbf{Rel}_f^{\mathcal{M}}$ where $p \langle \mathcal{U} \Downarrow \mathcal{V} \rangle = \mathcal{U}$ and $q \langle \mathcal{U} \Downarrow \mathcal{V} \rangle = \mathcal{V}$.

$$\begin{array}{ccc} & \text{minsnc}(f, g) & \\ p \swarrow & & \searrow q \\ A & & B \\ f \searrow & & \swarrow g \\ & X & \end{array} \quad (6)$$

4.2 Weak pullbacks in $\mathbf{Rel}_f^{\mathcal{M}}$

The following result shows that any synchronisation can be written as a linear combination of minimal synchronisations.

Lemma 42 *If $\langle \mathcal{U} \curlyvee \mathcal{V} \rangle$ then there exists a family $\{(k_i, \langle \mathcal{U}_i \curlyvee \mathcal{V}_i \rangle)\}_{i \in I}$, where each $\langle \mathcal{U}_i \curlyvee \mathcal{V}_i \rangle$ is minimal and different from $\langle \mathcal{U}_j \curlyvee \mathcal{V}_j \rangle$ for all $j \neq i$, s.t. $\langle \mathcal{U} \curlyvee \mathcal{V} \rangle = \sum_i k_i \langle \mathcal{U}_i \curlyvee \mathcal{V}_i \rangle$. The family is called a minimal decomposition of $\langle \mathcal{U} \curlyvee \mathcal{V} \rangle$.*

Proof. Simple induction. \square

The conclusion of Lemma 42 implies that (6) is a weak pullback diagram: given $\alpha : Y \rightrightarrows A$ and $\beta : Y \rightrightarrows B$ such that $f\alpha = g\beta$ in $\mathbf{Rel}_f^{\mathcal{M}}$, $h : Y \rightrightarrows \text{minsnc}(f, g)$ takes y to a minimal decomposition of $\langle \alpha y \curlyvee \beta y \rangle$.

Remark 43 *The diagram (6) is merely a weak pullback, because the decomposition of Lemma 42 is not, in general, unique. Indeed, consider $t : 2 \rightarrow 1$ in $\mathbf{Rel}_f^{\mathcal{M}}$ with $t_0 = t_1 = \{0\}$. Now $\text{minsnc}(t, t) = \{\langle \{0\} \curlyvee \{0\} \rangle, \langle \{0\} \curlyvee \{1\} \rangle, \langle \{1\} \curlyvee \{0\} \rangle, \langle \{1\} \curlyvee \{1\} \rangle\}$. Consider $u : 1 \rightarrow 2$ in $\mathbf{Rel}_f^{\mathcal{M}}$ with $u_0 = \{0, 1\}$. Then $\langle u_0 \curlyvee u_0 \rangle$ but there are several minimal decompositions: eg $\langle \{0\} \curlyvee \{0\} \rangle + \langle \{1\} \curlyvee \{1\} \rangle$ and $\langle \{0\} \curlyvee \{1\} \rangle + \langle \{1\} \curlyvee \{0\} \rangle$.*

5 Linking diagrams in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$

Consider $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$, with objects that the natural numbers and arrows spans $k \xleftarrow{f} x \xrightarrow{g} l$ in $\mathbf{Rel}_f^{\mathcal{M}}$ where $x \rightarrow k \times l$ is injective⁶. Composition proceeds in two steps. First, given

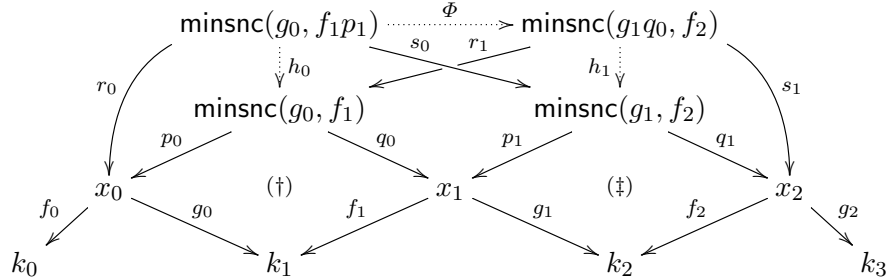
$$k_0 \xleftarrow{f_0} x_0 \xrightarrow{g_0} k_1 \xleftarrow{f_1} x_1 \xrightarrow{g_1} k_2,$$

construct $k_0 \xleftarrow{f_0 p} \text{minsnc}(g_0, f_1) \xrightarrow{g_1 q} k_2$ where $p : \text{minsnc}(g_0, f_1) \rightarrow x_0$ and $q : \text{minsnc}(g_0, f_1) \rightarrow x_1$ are the projections. In general, however, $[f_0 p, g_1 q] : \text{minsnc}(g_0, f_1) \rightarrow k_0 \times k_2$ may be non-injective, thus we obtain $\text{minsnc}(g_0, f_1)'$ below, together with f', g' in $\mathbf{Rel}_f^{\mathcal{M}}$ through an epi-mono factorisation of $[f_0 \# p, g_1 \# q]$ in \mathbf{Set} , and this is the composition.

$$k_0 \xleftarrow{f'} \text{minsnc}(g_0, f_1)' \xrightarrow{g'} k_2$$

Proposition 1. $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$ is a category.

Proof. (Sketch) The non-trivial part is showing that composition is associative. The essence is captured in the diagram below, in $\mathbf{Rel}_f^{\mathcal{M}}$.



⁶ In other words, the internal binary relations in $\mathbf{Rel}_f^{\mathcal{M}}$: an internal relation is a span $k \leftarrow x \rightarrow l$ where $x \rightarrow k \times l$ is mono.

In addition to the two weak pullback diagrams (\dagger) and (\ddagger), we have a set $\text{minsnc}(g_0, f_1 p_1)$ and the projection maps in $\mathbf{Rel}_f^{\mathcal{M}}$

$$r_0 : \text{minsnc}(g_0, f_1 p_1) \rightarrow x_0, \quad s_0 : \text{minsnc}(g_0, f_1 p_1) \rightarrow \text{minsnc}(g_1, f_2)$$

and a set $\text{minsnc}(g_1 q_0, f_2)$ together with maps

$$r_1 : \text{minsnc}(g_1 q_0, f_2) \rightarrow \text{minsnc}(g_0, f_1), \quad s_1 : \text{minsnc}(g_1 q_0, f_2) \rightarrow x_2$$

The sets $\text{minsnc}(g_0, f_1 p_1)$, $\text{minsnc}(g_1 q_0, f_2)$ are not, in general isomorphic, for similar reasons why the $\text{minsnc}(f, g)$ construction fails to be a pullback; there is, in general, more than one decomposition of a synchronisation into a linear combination of minimal synchronisations.

This is not a problem, because all that we require is that $(f_0 r_0, g_2 q_1 s_0)$ and $(f_0 p_0 r_1, g_2 s_1)$ have the same image in $\mathcal{M}_{k_0} \times \mathcal{M}_{k_2}$.

To show this, first we use the weak pullback property of (\dagger) to obtain $h_0 : \text{minsnc}(g_0, f_1 p_1) \rightarrow \text{minsnc}(g_0, f_1)$, satisfying $p_0 h_0 = r_0$ and $q_0 h_0 = p_1 s_0$. The second of these equations, together with the fact that $\text{minsnc}(g_1 q_0, f_2)$ is a weak pullback allows us to obtain

$$\Phi : \text{minsnc}(g_0, f_1 p_1) \rightarrow \text{minsnc}(g_1 q_0, f_2)$$

that satisfies $r_1 \Phi = h_0$ and $s_1 \Phi = q_1 s_0$. Now, for any $\sigma \in \text{minsnc}(g_0, f_1 p_1)$ we have $f_0 r_0 \sigma = f_0 p_0 h_0 \sigma = f_0 p_0 r_1 \Phi \sigma$ and $g_2 q_1 s_0 \sigma = g_2 s_1 \Phi \sigma$, so the image of $(f_0 r_0, g_2 q_1 s_0)$ is contained in the image of $(f_0 p_0 r_1, g_2 s_1)$. A symmetric argument, constructing morphisms $h_1 : \text{minsnc}(g_1 q_0, f_2) \rightarrow \text{minsnc}(g_1, f_2)$ and $\Psi : \text{minsnc}(g_1 q_0, f_2) \rightarrow \text{minsnc}(g_0, f_1 p_1)$ allows us to demonstrate the reverse inclusion. □

Note that, as indicated in the proof above, the ‘‘relational’’ requirement on spans is necessary in order to ensure associativity of composition. Again there is a tensor product inherited from the coproduct in \mathbf{Set}_f .

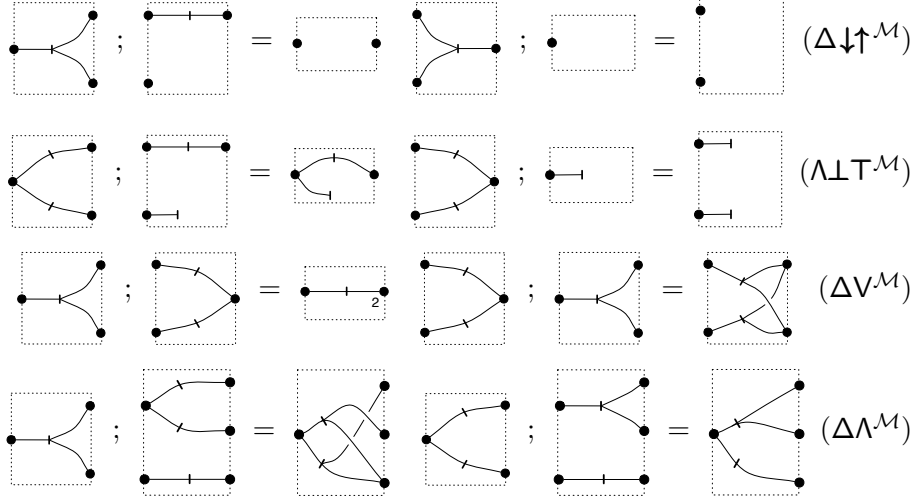
5.1 The algebra of $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$

While we no longer have to draw contention, in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$ links can have multiple connections to boundary ports. We indicate this by annotating connections with natural numbers ≥ 2 : for instance the diagram to the right is the span $2 \overset{a}{\dashv} 1 \overset{b}{\dashv} 2$ where $(a0)(0) = (b0)(1) = 1$, $(a0)(1) = 5$ and $(b0)(0) = 2$.

Considering the diagrams of $(\Delta \perp \nabla \top)$ and (IX) in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$, all the equations in (ΔUC) , (ΔA) , (F) , (S) , (CC) hold in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$. On the other hand, the structure in $(\Lambda \downarrow \vee \uparrow)$ and (IX) satisfies the equations in (ΛUC) , (ΛA) , (B) and $(\vee \downarrow)$. Differently from $(\Lambda \vee)$, in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$ we have the following:

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \diagup \quad \diagdown \\ \bullet \text{---} \end{array} ; \begin{array}{c} \text{---} \bullet \text{---} \\ \diagdown \quad \diagup \\ \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \bullet \text{---} \end{array} \quad (\Lambda \vee^{\mathcal{M}})$$

Below, we show how $(\Delta \perp \nabla \top)$ and $(\Lambda \downarrow \vee \uparrow)$ interact in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$.



The equations in $(\Delta\downarrow\uparrow^{\mathcal{M}})$ are the same as in $(\Delta\downarrow\uparrow^c)$. The left equation in $(\Lambda\perp\top^{\mathcal{M}})$ is the same as the corresponding one in $(\Lambda\top\perp^c)$, but the right hand side equations differ because the contention relation does not play a role in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$. The right hand side equation in $(\Delta\Lambda^{\mathcal{M}})$ agrees with the corresponding one in $(\Delta\mathbf{V}^c)$, but the left one deserves attention: while in $(\Delta\mathbf{V}^c)$ there was no possible synchronisation between Δ and \mathbf{V} because of the fact that the two links in \mathbf{V} were in contention, in $\mathbf{Sp}(\mathbf{Rel}_f^{\mathcal{M}})$ there is a synchronisation that involves all three links, as represented in the left hand side equation of $(\Delta\Lambda^{\mathcal{M}})$. The interaction between Λ and Λ is as in $(\Delta\Lambda^c)$, and $(\Delta\Lambda^{\mathcal{M}})$ are the same as $(\Delta\Lambda^c)$.

Theorem 3. *Every arrow in $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$ decomposes into an expression consisting only of $\Delta, \perp, \nabla, \top, \Lambda, \downarrow, \mathbf{V}, \uparrow, \mathbb{I}, \mathbb{X}$, composed with $;$ and \otimes .*

Proof. Omitted.

6 Conclusion

We have studied two categories of linking diagrams. The first, $\mathbf{Sp}(\mathbf{Rel}_f^c)$, arose from the study of a compositional algebra of C/E nets, called C/E nets with boundaries. Indeed, the arrows of $\mathbf{Sp}(\mathbf{Rel}_f^c)$ are just C/E nets with boundaries, without places. The second, $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$ arose from the study of a compositional algebra of P/T nets, called P/T nets with boundaries. The arrows of $\mathbf{Spr}(\mathbf{Rel}_f^{\mathcal{M}})$ are P/T nets with boundaries, without places. These categories generalise previous work by Hughes [11].

Both categories are “expressive enough” to carry two different commutative monoid-comonoid structures on objects, one of which a separable Frobenius algebra, the other a commutative bialgebra. In both settings the interaction between the two structures is interesting and we have examined some of the phenomena that arise. Both categories are generated by the small number of basic components that witness the monoid-comonoid structures.

In future work a full axiomatisation will be presented, and the categories of linking diagrams will be shown to characterise the arrows of the

resulting free categories. The theory of PROPs [16] seems well adapted for expressing the relationship between the algebraic structures, as well as the complete algebras of C/E and P/T nets; Fiore and Campos [9] have recently used a similar setting to develop the algebra of dags.

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