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# Deriving Bisimulation Congruences: A 2-categorical Approach

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**Abstract**

We introduce G-relative-pushouts (GRPO) which are a 2-categorical generalisation of relative-pushouts (RPO). They are suitable for deriving labelled transition systems (LTS) for process calculi where terms are viewed modulo structural congruence. We develop their basic properties and show that bisimulation on the LTS derived via GRPOs is a congruence, provided that sufficiently many GRPOs exist. The theory is applied to a simple subset of CCS and the resulting LTS is compared to one derived using a procedure proposed by Sewell.

## 1 Introduction

Term rewriting is a cornerstone of sequential computation. The λ-calculus, for instance, is essentially a simple term rewriting system. Process calculi, which aim at modelling concurrent computation, may also be viewed as rewriting systems, where the rewrites, the so-called *reactions*, represent the systems' internal evolution. The setting is more complex, however, as terms are often quotiented by a non-trivial *structural congruence*, which is a relation expressing which different syntactic representations describe the same process.

Park's notion of *bisimulation* [17,12] underpins a multitude of operational preorders and equivalences which allow reasoning about concurrent processes modelled in a particular process calculus. These rely on the presence of a *labelled transition system* (LTS) which may be seen as a description how processes interact with their environment. A LTS is a description of what may

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be observed about processes, for this reason bisimulation is often called an *observational equivalence*. Such equivalences are most useful when they are congruences, as this allows equational reasoning and full substitutivity.

Reaction systems and LTS usually coexist. The former are more easily postulated, as they tend to describe systems behaviour directly by focusing on the interactions between different parts, and therefore correspond closely to the calculus designer's computational intuitions. Indeed, one can derive sensible process equivalences using reactions. One such approach is the *barbed congruence* [16], which descends naturally from the sole choice of a specific notion of observable (a "barb"). A related approach is [4] which, based on intuitions from the $\lambda$-calculus, builds equational theories from directly rewrites requiring no a priori specification of observables.

On the contrary, LTS are often intensional: they aim at describing observable behaviours in a compositional way and, therefore, their labels may not be immediately justifiable in operational terms. In other words, it may not be obvious to identify a "natural" LTS for a given process calculus, even when its semantics is well understood. For example there are two alternative LTS for the $\pi$-calculus [15], the early and the late version, each giving a different bisimulation equivalence. Furthermore, once a LTS is given, it is usually non trivial to prove that bisimulation is a congruence.

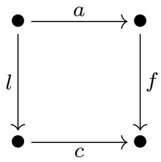
Due to the versatility of coinduction, LTS play a relevant role in applications. It is then important to be able to establish a correspondence between the two approaches. In particular, one may try to *synthesise* the LTS from the set of reactions. In a seminal work, Sewell [18] proposed several ways of doing this for restricted classes of term rewriting systems. The common idea is that certain contexts which allow reaction are taken as labels. Consider for instance the rewrite system consisting of the rule on the signature $\Sigma = \{a, b, c\}$ where $a$ is a unary function, $b$ and $c$ are constants and a single rewrite rule that transforms $ab$ into $c$, written $\langle ab, c\rangle$ . A possible LTS transition might then be

$$b \xrightarrow{a} c.$$

However, one usually takes only those contexts which are the *smallest* allowing a particular reaction to occur. As well as obtaining a LTS with fewer transitions, this often makes the resulting bisimulation equivalence finer: taking all the possible contexts as labels results in a bisimulation equivalence which is too coarse even in the case of very simple process calculi.

Sewell's method is based on dissection lemmas which provide a deep analysis of a term's structure, determining the missing triggers, if any. The proofs that bisimulation is a congruence on the resulting LTS is simple enough in the case of free syntax, but gets very complicated as soon as non trivial structural congruences are considered. Already in the case of the monoidal rules that govern parallel composition things become rather involved: the dissection method does not seem to scale to complex calculi.

A generalised approach was later developed in [10], where the notion of smallest is formalised in categorical terms as a *relative-pushout* (RPOs). Informally, consider a category in which arrows are terms and composition is substitution. In such a framework a context $f$ that allows $a$ to react according to rule $\langle l, r\rangle$ can be given as a commuting square:

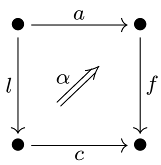



One derives a LTS by taking as labels those contexts $f$ which make such squares "minimal." The proof of congruence takes form as a theorem in pure category theory, requiring only the existence of "enough" RPOs.

Again, applying the theory of RPOs to "categories of terms" fails for process calculi with even simple structural congruences. One problem is that considering a commuting square like above when arrows are terms quotiented by a structural congruence, we lose too much information for the RPO approach to give the expected results. In particular, as we shall discuss in detail in §3, in a simple calculus with a parallel operator which is associative and commutative we lose the information of *where* in the term the reaction occurs. The indication of exactly which occurrences of a term constructor belong to the redex is fundamental in order to derive sensible LTS. The same problem arises when we replace syntactic terms by algebraic structures such as action graphs [13] and bigraphs [14]. Indeed, essentially because of the problem of locating reactions, sufficient RPOs usually do not exist [8,14].

For syntactic terms, Sewell proposes to deal with this by means of a notion of colouring [18], while Leifer [9] suggests an abstract approach via *functorial reactive systems* and *precategories*. We believe that the techniques presented here substantially simplify such a theory, and we offer a brief exposition in §6, although the details are left for future publication.

The approach proposed in this paper is to keep the information of "how the square commutes" by retaining the derivation of structural congruence. In this case, we think of the structural congruence as a set of rules governing how a term tree may be altered without changing the process (or in the algebraic case, as a set of suitable structure preserving isomorphisms). This information naturally gives a 2-categorical structure, where a 2-cell like the one below represents a "derivation" of the structural congruence of $fa$ and $cl$.



Indeed, this geometric point of view is close to the spatial metaphor in the Chemical Abstract Machine [1] which served as an inspiration for the use of structural congruences in the operational semantics of process calculi. Observe that, since the structural alteration of term trees is always reversible in this setting, in our categories all 2-cells are isomorphisms. For such *locally groupoidal 2-categories*, G-categories for short, we propose a suitable generalisation of RPO, the GRPO. We shall prove that they enjoy the fundamental properties of RPOs and, in particular, that bisimulation on a LTS derived using GRPOs is a congruence if sufficiently many GRPOs exist.

While addressing the same concern above about redex location, we claim that our proposal is significantly simpler and more general than previously published work.

**Structure of the paper.** In §2 we review RPOs and Sewell's derivation of LTS for ground rewriting systems on free syntax, illustrating the relationship between the two approaches. In §3 we show in detail why the RPO approach fails when terms are viewed modulo structural congruence, and why a 2-categorical approach may be desirable. In §4 we introduce and develop the theory of GRPOs and give the congruence proof. In §5 we apply the theory to derive LTS for a simple fragment of CCS and compare the results to those of Sewell's approach. Finally, in section §6 we conclude by offering possible directions for future work.

## 2 Relative Pushouts

In this section we give a brief review of the theory of RPOs, a more complete presentation may be found in [8]. We end with a comparison to the work of Sewell [18] for ground rewriting systems on free syntax.

Consider a signature $\Sigma$. The (free) Lawvere theory for $\Sigma$ [7], denoted as $\mathsf{Th}(\Sigma)$, is a category with objects natural numbers and morphisms $t\colon m \to n$ being $n$-tuples of $m$-holed terms. Composition is substitution of terms. The category is cartesian, with 0 the terminal object and $n$ being the product of 1 with itself $n$ times. Identities $n \to n$ are $\langle -_1, -_2, \ldots, -_n \rangle$. The theory is free in the sense that there are no equations between composite terms, apart from those imposed by the cartesian structure. A morphism $t\colon m \to n$ is linear if each of the $m$ "holes" is used exactly once in $t$. Let $\mathbf{C}_\Sigma$ denote the subcategory of $\mathsf{Th}(\Sigma)$ consisting of the linear morphisms.

A term rewriting system is can be given as a set $\mathcal{R}$ of pairs $\langle l, r\rangle$ where $l, r : n \to 1$ are arrows of $\mathbf{C}_\Sigma$. The reaction relation $\longrightarrow\!\!\!\!\triangleright$ is derived from $\mathcal{R}$ by substitution under contexts, that is $a \longrightarrow\!\!\!\!\triangleright\ a'$ if $a = cl$, $a' = cr$ for some $c \in \mathbf{C}_\Sigma$. A term rewriting system is a ground term rewriting system when $\mathcal{R}$ consists only of pairs $\langle l, r\rangle$ with $l, r : 0 \to 1$.

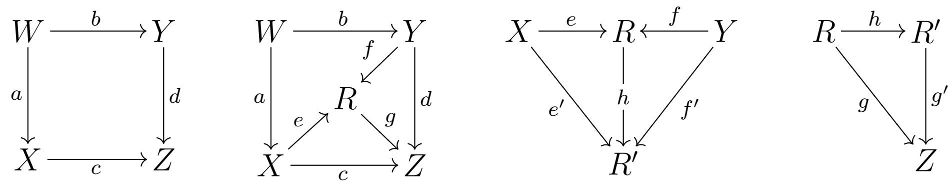
Generalising from ground term rewriting systems on $\mathbf{C}_\Sigma$, we give the definition of *reactive system* from [10].

**Definition 2.1 (Reactive System)** A reactive system $\mathbf{C}$ consists of a category $\mathbb{C}$, a composition-reflecting subcategory $\mathbb{D}$ of *reactive contexts*, a distinguished object $I \in \mathbb{C}$ and a set or pairs $\mathcal{R} \subseteq \bigcup_{C \in \mathbb{C}} \mathbb{C}(I, C) \times \mathbb{C}(I, C)$.

By composition-reflecting we mean that $dd' \in \mathbb{D}$ implies $d, d' \in \mathbb{D}$. The reactive contexts are those contexts inside which evaluation may occur. The reaction relation $\longrightarrow\!\!\triangleright$ is derived from $\mathcal{R}$ by closing it under all reactive contexts. For simplicity, we shall henceforward assume that all contexts are reactive, that is, $\mathbb{D} = \mathbb{C}$. This will be the case for all the examples mentioned in this paper, while the proof of congruence needs only to be altered slightly to accommodate $\mathbb{D}$.

The notion of RPO formalises the idea of a context being the "smallest" that enables a reaction in a reactive system.

**Definition 2.2 (RPO)** Let $\mathbf{C}$ be a reactive system and $(i)$ a commuting diagram in $\mathbf{C}$.



$(i)$ $(ii)$ $(iii)$ $(iv)$

Any tuple $\langle R, e, f, g \rangle$ which makes $(ii)$ commute is called a *candidate* for $(i)$. A relative pushout is the "smallest" such candidate. More formally, it satisfies the universal property that given any other candidate $\langle R', e', f', g' \rangle$, there exists a *unique* mediating morphism $h \colon R \to R'$ such that $(iii)$ and $(iv)$ are commuting.

Another way of viewing RPOs is as ordinary pushouts in a slice-category. Indeed, the commuting square $(i)$ above is simply a span

$$(X, c) \xleftarrow{\;a\;} (W, ca) \xrightarrow{\;b\;} (Y, d)$$

in the slice category $\mathbb{C}/Z$. It is straightforward to verify that to give a relative pushout of $(i)$ above is to give a pushout of the span in $\mathbb{C}/Z$.

**Definition 2.3 (IPO)** A commuting square like $(i)$ of Definition 2.2 is a idem-relative-pushout (IPO) if $\langle Z, c, d, \mathrm{id}_Z \rangle$ is its RPO.

For $\mathbf{C}$ a reactive system, a labelled transition system $\mathbf{TS}(\mathbf{C})$ can be derived using IPOs as follows:

- the states of $\mathbf{TS}(\mathbf{C})$ are arrows $a \colon I \to X$ of $\mathbf{C}$;

- there is a transition $a \xrightarrow{b} cr$ in $\mathbf{TS}(\mathbf{C})$ if and only if $\langle l, r \rangle \in \mathcal{R}$ and

$$\begin{array}{ccc} I & \xrightarrow{a} & X \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle b} \\ Y & \xrightarrow[c]{} & Z \end{array} \qquad \text{is an IPO.}$$

In other words, if insertion in context $b$ makes $a$ match $l$ in context $c$ (commutation of the diagram), where $l$ is a redex, and $b$ is the "smallest" such context (IPO condition), then $a$ moves to $cr$ with label $b$, where $r$ is the reduct of $l$.

A reactive system $\mathbf{C}$ is said to have redex RPOs if every commuting square $cl = ba$ in $\mathbb{C}$, where $\langle l, r \rangle \in \mathcal{R}$, has an RPO. If this condition is satisfied, then $\sim$, the largest bisimulation on $\mathbf{TS}(\mathbf{C})$, is a congruence [10]. This result is generalised in this paper to a 2-categorical notion of RPOs (cf. Theorem 4.10).

Often it is desirable to consider only terms $a$ of a fixed arity $I \to T$ and labels of type $T \to T$. Let $\mathbf{TS}(\mathbb{C})_T$ denote the labelled transition system so obtained and let $\sim_T$ be corresponding bisimulation. If $\mathbb{C}$ has redex RPOs, then it follows from the general proof that $\sim_T$ is also a congruence. Clearly, $\sim \subseteq \sim_T$, and the converse does not hold in general.

For ground term rewriting on $\mathbf{C}_\Sigma$, Sewell derives a LTS $\mathbf{Sew}(\mathbf{C}_\Sigma)$ with nodes being terms $a : 0 \to 1$ and labels $f : 1 \to 1$ as follows:

- $s \xrightarrow{-} t$ iff $s \longrightarrow\!\rhd\, t$
- $s \xrightarrow{f} t$ iff there exists $\langle l, r \rangle \in \mathcal{R}$ such that $fs = l$ and $r = t$ (for $f \neq -$).

The two definitions are related. Indeed, using dissections [18], one can prove that redex RPOs exist in $\mathbf{C}_\Sigma$ [19]. The following Lemma is due to Sewell [19].

**Lemma 2.4** $\mathbf{TS}(\mathbf{C}_\Sigma)_1 = \mathbf{Sew}(\mathbf{C}_\Sigma)$.

***Proof.*** It suffices to show that

$$\begin{array}{ccc} 0 & \xrightarrow{a} & 1 \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle c} \\ 1 & \xrightarrow[d]{} & 1 \end{array}$$

is an IPO iff either $c = -$ or $d = -$. Indeed, suppose that $d \neq -$ and $c \neq -$. Then $d$ and $c$, viewed as term trees, contain a topmost node labelled by $\sigma : n \to 1$ where $\sigma \in \Sigma$. This $\sigma$ can be used to construct a non-trivial candidate for the diagram above, contradicting the assumption that the square is an IPO. □

## 3 Structural Congruence

In this section we discuss the motivation for considering a notion of relative-pushout in a 2-categorical setting.

**Example 3.1** Consider the following simple subset of CCS:

$$P \ ::= \ \mathbf{0} \ \mid \ a \ \mid \ \overline{a} \ \mid \ P \,|\, P' \qquad\qquad \text{where } a \in N.$$

The signature consists of constants symbols for channel names and for the null process and a binary operator, that is $\Sigma = \{\mathbf{0}, a, \overline{a}, -_1 \,|\, -_2, \}$. The reaction relation is the closure of the relation $\{\, (a \,|\, \overline{a}, \mathbf{0}) \mid a \in N \,\}$ under all contexts. The standard operational semantics can be summarised by the following rules,

$$a \xrightarrow{a} \mathbf{0} \qquad\qquad \overline{a} \xrightarrow{\overline{a}} \mathbf{0} \qquad\qquad a \,|\, \overline{a} \xrightarrow{\tau} \mathbf{0}$$

$$\frac{P \xrightarrow{x} P'}{Q \,|\, P \xrightarrow{x} Q \,|\, P'} \qquad\qquad \frac{P \equiv P' \quad P \xrightarrow{x} Q \quad Q' \equiv Q}{P' \xrightarrow{x} Q'}$$

where $\equiv$ is the smallest congruence on the set of terms over $\Sigma$ which makes $|$ an action of a commutative monoid with $\mathbf{0}$ as identity.

Let $\mathbf{D}_\Sigma$ be a category with the same objects as $\mathbf{C}_\Sigma$ but whose arrows are terms quotiented by $\equiv$. One may ask what happens if one uses the RPO approach to generate an LTS. Consider the term $a \,|\, \overline{a}$. Using the standard operational semantics we should expect three transitions,

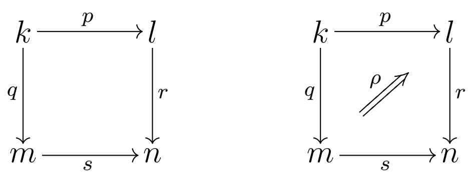
$$a \,|\, \overline{a} \xrightarrow{a} \overline{a}, \qquad a \,|\, \overline{a} \xrightarrow{\overline{a}} a \qquad \text{and} \qquad a \,|\, \overline{a} \xrightarrow{\tau} \mathbf{0}.$$

Consider the three squares in $\mathbf{D}_\Sigma$ below, where we use subscripts to distinguish different occurrences of the term $a$ (that may float around in larger terms because of $\equiv$). Observe that such distinction is for the sake of exposition only: arrows in $\mathbf{D}_\Sigma$ up to structural congruence, and therefore individual occurrences of terms are not discernible.

$$\begin{array}{ccc} 0 & \xrightarrow{a_1|\overline{a}_2} & 1 \\ {\scriptstyle a_1|\overline{a}_2}\downarrow & & \downarrow{\scriptstyle -} \\ 1 & \xrightarrow[-]{} & 1 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{a_1|\overline{a}_2} & 1 \\ {\scriptstyle \overline{a}_2|a_3}\downarrow & & \downarrow{\scriptstyle -|a_3} \\ 1 & \xrightarrow[a_1|-]{} & 1 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{a_1|\overline{a}_2} & 1 \\ {\scriptstyle \overline{a}_3|a_1}\downarrow & & \downarrow{\scriptstyle \overline{a}_3|-} \\ 1 & \xrightarrow[-|\overline{a}_2]{} & 1. \end{array}$$

Only the left one could possibly be an IPO, and it is easy to see that it is a *candidate* for the middle and the right squares. Indeed, since the term and the left hand side of the reaction rule are identical, the identity context is clearly the smallest "upper bound." However, also the upper bounds given in the middle and the right square are in some sense *minimal*. Indeed, if one keeps track of the place in the term where the reaction occurs, then the middle square is the smallest upper bound whose redex (viz. $\overline{a}_2 \,|\, a_3$) *only* uses $\overline{a}$ (as opposed to both $a$ and $\overline{a}$) from the term. Similarly, in the right square the redex created by insertion into a context (viz. $\overline{a}_3 \,|\, -$) only uses $a$. It is precisely the fact that terms in $\mathbf{D}_\Sigma$ are quotiented by $\equiv$ that makes it impossible to place reaction within a term.

At this point it is important to focus on what exactly is a commuting square in $\mathbf{D}_\Sigma$. To verify that a diagram like $(i)$ below is commuting one has to exhibit a proof of structural congruence constructed from the basic rules closed under all contexts.



$(i)$ $(ii)$

Different proofs can be chosen to exhibit commutativity. Indeed, with a bit of massaging, such "proofs" can be represented as 2-cells and used to give a 2-categorical structure on $\mathbf{C}_\Sigma$.

Along with a suitable 2-categorical notion of a relative-pushout we get a natural notion of "smallest" which remembers the location of the redex and that for the example above works much the same way as Sewell's colouring of terms [18]. However, since our definition is abstract, it also gives a way of approaching process calculi with structural congruences different from $(\mid, \mathbf{0})$.

In the following section we give the details of our generalisation of RPOs to 2-categories and we show that they enjoy the congruence properties of their 1-dimensional cousins.

## 4 2-categories and GRPOs

For an introduction to 2-categories we refer the reader to [6]. For the reader's convenience we recall some basic definitions in Appendix A. We shall denote horizontal composition of 2-cells by juxtaposition and vertical composition by '$\bullet$'. Horizontal composition binds tighter than vertical.

In the following we will be concerned with the class of 2-categories whose all 2-cells are isomorphisms, i.e. invertible. Referring to categories whose arrows are all invertible as groupoids, this class can be identified with the class of groupoid-enriched categories.

**Definition 4.1 (G-Category)** A G-category is a category enriched over G, the category of groupoids.

**Example 4.2** Consider the subset of CCS introduced in Example 3.1.

Let $\mathbf{M}_\Sigma$ be the G-category with:

- a single object $I$;
- arrows strings $a_1 \mid a_2 \mid \dots \mid a_n$, $a_i \in N$ with composition by juxtaposition (eg. $(a_3 \mid a_4)(a_1 \mid a_2) = a_1 \mid a_2 \mid a_3 \mid a_4$) and the empty string denoted by $\mathbf{0}$ serving as the identity;
- 2-cells permutations; namely, each arrow $a_1 \mid a_2 \mid \dots \mid a_n$ is the source

of $n!$ 2-cells determined by the permutations $\varphi\colon [n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$. Each such $\varphi$ determines a 2-cell

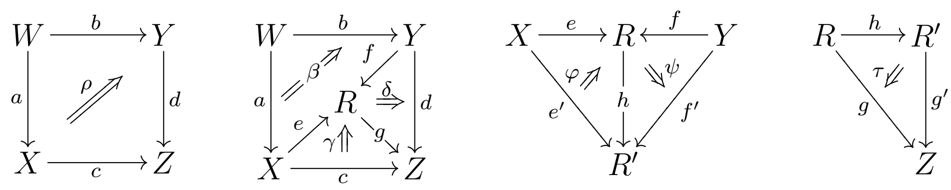
$$\varphi_{a_1,a_2,\ldots,a_n} \colon a_1 \mid a_2 \mid \cdots \mid a_n \Rightarrow a_{\varphi^{-1}(1)} \mid a_{\varphi^{-1}(2)} \mid \cdots \mid a_{\varphi^{-1}(n)}.$$

For clarity we will usually leave out the subscripts. So, for example, there are two 2-cells $a \mid a \Rightarrow a \mid a$: the identity, and the permutation that "swaps" the two occurrences of $a$. Vertical composition is via composition of permutations, horizontal composition is via "juxtaposition," i.e. for $\varphi\colon [m] \to [m]$ and $\psi\colon [n] \to [n]$, we define $\psi\varphi\colon [m+n] \to [m+n]$ by $(\psi\varphi)(i) = \varphi(i)$ for $i \leq m$ and $(\psi\varphi)(i) = m + \psi(i - m)$ for $i > m$.

It should be clear to the reader that $p \equiv q$ iff there exists a 2-cell $\rho\colon p \Rightarrow q$.

We now present a generalisation of the notion of RPO to G-categories.

**Definition 4.3 (GRPO)** Let $\mathbb{C}$ be a G-category. A candidate (GRPO) for *(i)* below




*(i)* *(ii)* *(iii)* *(iv)*

is a tuple $\langle R, e, f, g, \beta, \gamma, \delta\rangle$ such that $\delta b \bullet g\beta \bullet \gamma a = \rho$ (cf. diagram (*ii*)).

A G-relative-pushout (GRPO) for *(i)* is a candidate which satisfies a universal property, namely, for any other candidate $\langle R', e', f', g', \beta', \gamma', \delta'\rangle$ there exists a quadruple $\langle h, \varphi, \psi, \tau\rangle$ where $h\colon R \to R'$, $\varphi\colon e' \Rightarrow he$ and $\psi\colon hf \Rightarrow f'$ (cf. diagram (*iii*)) and $\tau\colon g'h' \Rightarrow g$ (diagram (*iv*)) which makes the two candidates compatible in the obvious way.

Spelling this out, the equations that need to be satisfied are:

(i) $\tau e \bullet g'\varphi \bullet \gamma' = \gamma$;

(ii) $\delta' \bullet g'\psi \bullet \tau^{-1} f = \delta$;

(iii) $\psi b \bullet h\beta \bullet \varphi a = \beta'$.

We shall refer to such a quadruple as a *mediating morphism*. Such a morphism must be *essentially unique*, namely, for any other mediating morphism $\langle h', \varphi', \psi', \tau'\rangle$ there must exist a unique two cell $\xi\colon h \to h'$ which makes the two mediating morphisms compatible, i.e.:

(i) $\xi e \bullet \varphi = \varphi'$;

(ii) $\psi \bullet \xi^{-1} f = \psi'$;

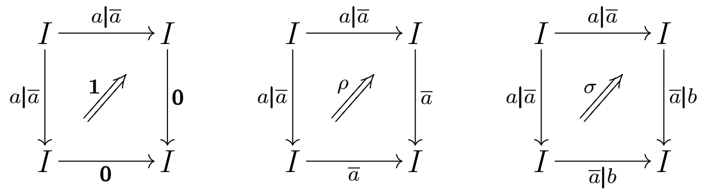
(iii) $\tau' \bullet g'\xi = \tau$.

Whereas RPOs are defined up to isomorphism, GRPOs are defined up to equivalence.

It is worth noticing that the above definition is a simplification to G-categories of a notion of biRPO on 2-categories, defined as a bipushout in a suitable pseudo-slice-category (just as a RPO is a pushout in a slice category). For details we refer the reader to Appendix A. Clearly, biRPOs (and therefore GRPOs) generalise RPOs: if one considers a category as a discrete 2-category (the only 2-cells are identities) then a biRPO is simply a RPO.

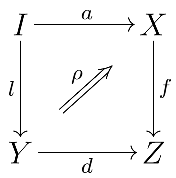
**Definition 4.4 (GIPO)** Diagram $(i)$ of Definition 4.3 is said to be a G-idem-pushout (GIPO) if $\langle Z, c, d, \mathrm{id}_Z, \rho, 1_c, 1_d\rangle$ is its GRPO.

**Example 4.5** Consider the category $\mathbf{M}_\Sigma$ from Example 4.2.



Consider the three squares above, where $\rho(2) = \sigma(2) = 3$ and $\rho(3) = \sigma(3) = 2$. Informally, the two copies of $\overline{a}$ are swapped in the middle and the right squares. The first two squares are GIPOs, but the third square is not. We leave the proofs to the reader.

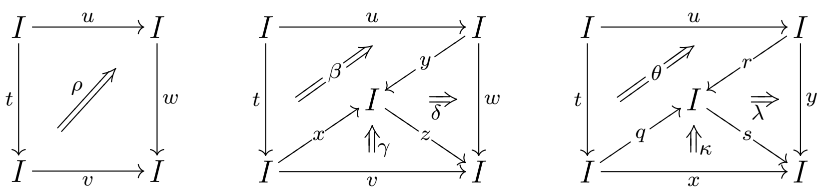
**Definition 4.6 (Redex GRPOs)** A reactive system $\mathbf{C} = \langle \mathbb{C}, \mathbb{D}, \mathcal{R}, I\rangle$ where $\mathbb{C}$ is a G-category is said to have redex-GRPOs if every square



(1)

where $l$ is the left hand side of a reaction rule $\langle l, r\rangle \in \mathcal{R}$ has a GRPO.

**Example 4.7** The category $\mathbf{M}_\Sigma$ from Example 4.2 has redex GIPOs for any choice of $\mathcal{R}$. Indeed, consider any square *(i)* as below



$(i)$ $(ii)$ $(iii)$

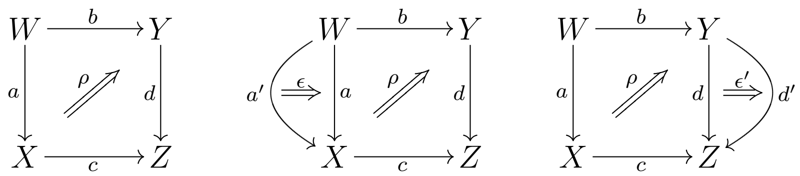
where $t = t_1 \mid \cdots \mid t_{|t|}$, $u = u_1 \mid \cdots \mid u_{|u|}$, $v = v_1 \mid \cdots \mid v_{|v|}$ and $w = w_1 \mid \cdots \mid w_{|w|}$. We use $|-|$ to count the number of parallel components in terms. Then take $z = v_{\ell_1} \mid \cdots \mid v_{\ell_k}$ and $x = v_{\ell_{k+1}} \mid \cdots \mid v_{\ell_{|v|}}$, where $\{\ell_1, \ldots, \ell_k\} + \{\ell_{k+1}, \ldots, \ell_{|v|}\}$ is a partition of $[|v|]$ such that $\rho(\ell_j + |t|) \leq |l|$

iff $j > k$. Similarly, let $y = w_{m_1} \mid \cdots \mid w_{m_h}$, for $\{m_1, \ldots, m_h\}$ the subset of $[|w|]$ such that $\rho^{-1}(m_j) < |t|$. Then $\gamma$, $\delta$ and $\beta$ are uniquely determined so that $\delta u \bullet z\beta \bullet \gamma t = \rho$, as illustrated in *(ii)*. Note that $\beta(i + |t|) \leq |u|$ for $i \geq 1$.$^{(*)}$ We claim that the resulting square *(iii)* is a GIPO. Indeed, suppose that $\langle q, r, s, \theta, \kappa, \lambda \rangle$ is a candidate. By $(*)$, $s = \mathbf{0}$ and $\langle \mathbf{0}, \kappa^{-1}, \lambda^{-1}, 1_{\mathbf{0}} \rangle$ is the unique mediating morphism.

Notice that in fact *(i)* is a GIPO if and only if $(*)$ above holds.

The next lemma proves one of the fundamental properties of GRPOs.

**Lemma 4.8** Suppose that diagram *(i)* below is a GIPO and than $\epsilon \colon a' \Rightarrow a$, $\epsilon' \colon d \to d'$ are isomorphisms.




*(i)* *(ii)* *(iii)*

Then the regions obtained by pasting the 2-cells in *(ii)* and *(iii)* are GIPOs.

**Proof.** See Appendix B. □

**Definition 4.9 (LTS)** For $\mathbf{C}$ a reactive system whose underlying category $\mathbb{C}$ is a G-category, define $\mathbf{GTS}(\mathbf{C})$ as follows:

- the states $\mathbf{GTS}(\mathbf{C})$ are iso-classes of arrows $[a] \colon I \to X$ in $\mathbf{C}$
- there is a transition $a \xrightarrow{[f]} a'$ if there exists a 2-cell $\alpha$, $\langle l, r \rangle \in \mathcal{R}$ and $d$ in $\mathbb{B}$ such that Diagram 1 is a GIPO and $a' = dr$.

Observe that the LTS is well defined by Lemma 4.8.

**Theorem 4.10** *Let* $\mathbf{C}$ *be a reactive system whose underlying* G*-category* $\mathbb{C}$ *has redex* G*RPOs. The largest bisimulation* $\sim$ *on* $\mathbf{GTS}(\mathbf{C})$ *is a congruence.*

The proof is essentially the same as the one given in [8] for RPOs. It is given in Appendix C together with supporting lemmas (which are interesting in their own right).

## 5 Comparison with Colouring

Sewell proposed an elegant derivation of LTS for ground term rewriting systems on syntax containing $\{\mid, \mathbf{0}\}$ where terms are viewed modulo the standard structural congruence rules. The derivation procedure uses the notion of *colouring* [18]. We shall briefly recall the details and compare the LTS derived with the one derived using the theory of GRPO for the simple calculus of Example 4.2 (see Example 4.5 for sample labels). The reader should note that Sewell considers arbitrary signatures $\Sigma$ with $\{\mid, \mathbf{0}\}$ and the relevant structural congruence; here we only consider signatures $\Sigma$ with $\{|, \mathbf{0}\}$ and constants. The GIPO approach can be extended to arbitrary signatures by adopting a suitable 2-categorical extension of linear Lawvere theories (cf. Section 2). Such structures are called Lawvere 2-theories and have been used, e.g., by Meseguer [11] to provide presentation-independent realisations of rewrite theories. We plan to pursue this direction in future work.

Let $\{|, \mathbf{0}\} \subseteq \Sigma$. Let $C = \{\mathit{red}, \mathit{blue}\}$ be a set of colours and let $\Sigma^C$ denote the coloured signature, it consists of $\{|, \mathbf{0}\}$ and coloured symbols $\sigma^c$, $c \in \mathbb{C}$, $\sigma \notin \{|, \mathbf{0}\}$.

Let $M_{\Sigma^c}$ and $M_\Sigma$ denote the categories constructed as in Example 4.2. There is an obvious "underlying symbol" 2-functor $|-|\colon M_{\Sigma^C} \to M_\Sigma$. There are also 2-functors $(-)^{\mathit{red}}\colon M_\Sigma \to M_{\Sigma^C}$ and $(-)^{\mathit{blue}}\colon M_\Sigma \to M_{\Sigma^C}$ which colour non $\{|, \mathbf{0}\}$ symbols red and blue respectively.

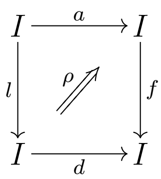
**Definition 5.1** Define a labelled transition system $\mathbf{Sew}^c(M_\Sigma)$ as follows

- The nodes are elements $a \in M_\Sigma$;
- $s \xrightarrow{f} t$ iff there exists $\langle l, r\rangle \in \mathcal{R}$, $f^{\mathit{red}}\mathbf{s} \equiv d^{\mathit{blue}} l^{\mathit{red}}$, $|\mathbf{s}| = s$ and $t \equiv dr$

Intuitively, $f$ contains only information necessary for the reaction.

**Theorem 5.2** $\mathbf{GTS}(M_\Sigma) = \mathbf{Sew}^c(M_\Sigma)$.

***Proof.*** It suffices to show that there exists a 2-cell $\rho$ such that the diagram below is a GIPO if and only if there exists a colouring $\mathbf{a}$ of $a$ such that $f^{\mathit{red}}\mathbf{a} = d^{\mathit{blue}} l^{\mathit{red}}$.




Recall from Example 4.7 that such a square is a GIPO if and only if $\rho(i+|l|) \leq |a|$ for $0 < i \leq |d|$.

Suppose that the square is a GIPO. Assume that $a$, $l$, $f$ and $d$ are coloured red. Certainly $f^{\mathit{red}} a^{\mathit{red}} \equiv d^{\mathit{red}} l^{\mathit{red}}$, as exhibited by $\rho$. We show that $d$ can be coloured blue while not changing the colour of $f$. Indeed supposing that $d = d_1 \mid \ldots \mid d_k \mid \ldots \mid d_{|d|}$, we have $\rho(k + |l|) \leq |a|$ and we can colour $d_k$ and its image under $\rho$ blue as the image lies in $a$.

Now assume that $f^{\mathit{red}}\mathbf{a} \equiv d^{\mathit{blue}} l^{\mathit{red}}$ and let $\rho$ be a two-cell which exhibits this equivalence. Suppose that $\langle q, r, s, \theta, \kappa, \lambda\rangle$ is a candidate. Then $s$ consists of elements which are both in $f$ and in $d$. Since $f$ and $d$ are monochrome and differ in colour, $s = \mathbf{0}$. This implies that $\rho(i + |l|) \leq |a|$ for $0 < i \leq |d|$ and therefore the square is a GIPO. □

## 6 Conclusion and Future Work

We have presented the theory of G-relative-pushouts, a generalisation of Leifer and Milner's relative-pushouts to locally groupoidal 2-categories (G-categories) The theory allows derivation of labelled transition systems which are automatically congruences under certain general conditions. The novelty of the approach is that, by keeping track of the application of structural congruence rules (as 2-cells), we are able to derive more informative labelled transition systems. We have demonstrated an application to a simple calculus with an associative and commutative parallel operator. Work is underway to apply the theory in the presence of complex structural congruences, in particular replication. We hope that eventually this research will lead to a uniform treatment of an interesting class of process calculi. We envisage that such an approach may be based on suitable Lawvere 2-theories of calculi, as mentioned briefly in §5.

Moving away from syntax based reactive systems, our 2-categorical approach could prove useful when syntactic terms are replaced by algebraic objects, such as graphs, action graphs [13] or bigraphs [14]. In such cases the 2-cells would be suitable structure preserving isomorphisms.

A simple example is the category of bunches, as considered in [10]. By taking the 2-cells as permutations of the leaves, one can specify bunches elegantly, leaving out the so-called "trailing" data. We claim that GRPOs give the same LTS on such simpler bunches as RPOs do on the original definition.

The synthesis of LTS for action graphs and bigraphs relies on the *functorial reactive systems*, introduced by Leifer [8]. They feature a category "above" related to the category of interest via a functor and decorated with trailing information so as to guarantees enough RPOs. Labels are derived accordingly and the LTS enjoys the expected congruence properties, under suitable conditions on the functor. Categories "above" and the corresponding functors can usually be generated automatically from so-called *precategories*. As foreseen by Leifer [9], such a framework can be "compressed" to a bicategorical notion of RPO, where the bicategory carries the same information as the precategory. We believe that GRPOs generate in this new setting the same LTS as the functorial reactive systems do for action graphs and bigraphs, and we claim that they provide a significantly simpler and more flexible approach. (Notice that we need not restate the theory of GRPOs for bicategories, as every bicategory is equivalent to a 2-category [21].)

The notion of GRPO seems also natural in the context of graph transformation systems (GTS) [2] realised as graph cospans, similarly to [3]. Applying the theory of GRPO in this setting will provide, we hope, interesting new LTS-based semantic theories for GTS.

**Acknowledgement.** The authors would like to thank Peter Sewell for generously providing access to his unpublished working notes and Steve Lack for pointing out useful references and for many helpful comments on categorical aspects of this work. Thanks also goes to the referees for their helpful suggestions.

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## A 2-categories etc.

Here we recall the basic definitions. A 2-category is a Cat-enriched category, that is a category whose homset and related composition maps live in the category of (small) categories. In more explicit terms, a 2-category $\mathbb{B}$ consists of what follows.

- A class of objects $X, Y, Z, \ldots$.
- For any $X, Y \in \mathbb{C}$, a category $\mathbb{C}(X, Y)$. The objects $\mathbb{C}(X, Y)$ are called 1-cells, or simply arrows, and denoted by $f\colon Y \to X$. Its morphisms are called 2-cells, and written as $\alpha\colon f \Rightarrow g$. Composition in $\mathbb{C}(X, Y)$ is denoted by $\bullet$ and referred to as 'vertical' composition. Identity 2-cells are denoted by $\mathbf{1}_f\colon f \Rightarrow f$.
- For each $X, Y, Z$ there is a functor $\circ\colon \mathbb{C}(X, Y) \times \mathbb{C}(Y, Z) \to \mathbb{C}(X, Z)$, the so-called 'horizontal' composition, we shall usually denote horizontal composition by juxtaposition. Horizontal composition is associative and admits $\mathbf{1}_{\mathrm{id}_X}$ as identities. Note that the functoriality of  can be spelt out as: for all $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$, $\gamma\colon u \Rightarrow v$, $\delta\colon v \Rightarrow w$, where $f, g, h\colon X \to Y$ and $u, v, w\colon Y \to Z$, we have $(\delta \bullet \gamma)(\beta \bullet \alpha) = \delta\beta \bullet \gamma\alpha$ and $\mathbf{1}_g.\mathbf{1}_f = \mathbf{1}_{gf}$. As a matter of notation, we write $\alpha f$ and $g\alpha$ for, respectively, $\alpha\mathbf{1}_f$ and $\mathbf{1}_g\alpha$.

Given a 2-category $\mathbb{C}$ we shall sometimes refer to the underlying category $\mathbb{C}_1$, this is the category obtained from $\mathbb{C}$ by forgetting the 2-cells.

**Definition A.1 (2-Category)** A 2-category is a category enriched over Cat.

Just as a RPO is a pushout in a slice category, a GRPO is a bipushout in pseudo-slide category $\mathbf{C}/X$.

**Definition A.2 (Bipushout)** A bipushout of arrows $X \xleftarrow{a} W \xrightarrow{b} Y$ is a quadruple $\langle Z, c, d, \rho\rangle$ where $c\colon X \to Z$, $d\colon Y \to Z$ and $\rho\colon ca \Rightarrow db$ is an isomorphism such that, for any other such quadruple $\langle Z', c', d', \rho'\rangle$:

(1) there exists an arrow $u\colon Z \to Z'$ and isomorphisms $\varphi\colon c' \Rightarrow uc$, $\psi\colon ud \Rightarrow d'$ satisfying the obvious compatibility condition, namely $\psi b \bullet u\rho \bullet \varphi a = \rho'$.

(2) For any other arrow $u'\colon Z \to Z'$ and 2-cells $\eta\colon u'c \Rightarrow uc$, $\mu\colon u'd \to ud$ satisfying $u\rho \bullet \eta a = \mu b \bullet u'\rho$, there exists a unique $\xi\colon u' \to u$ such that $\eta = \xi c$ and $\mu = \xi d$.

If the 2-category in question is a G-category then we can rewrite (2) as follows:

(2′) For any other triple $\langle u', \varphi', \psi' \rangle$ satisfying the equations of (1) there exists a unique $\xi\colon u' \Rightarrow u$ such that $\xi c \bullet \varphi' = \varphi$ and $\psi' \bullet \xi^{-1} d = \psi$.

Note that bipushouts are actually instances of bicolimits, as originally defined by Street [20] and also examined briefly by Kelly [5] in a 2-categorical setting

Given a 2-category $\mathbf{C}$ and an object $Z$, a pseudo-slice-category $\mathbf{C}/Z$ is a 2-category with objects $C \xrightarrow{f} Z$, arrows $(C, f) \xrightarrow{(h,\epsilon)} (D, g)$ where $h\colon C \to D$ and $\epsilon\colon f \Rightarrow gh$ is an isomorphism and 2-cells $\xi\colon (h, \epsilon) \Rightarrow (h', \epsilon')$ being 2-cells $\xi\colon h \to h'$ satisfying the obvious compatibility requirement, namely $g\xi \bullet \epsilon = \epsilon'$.

Analogously to the the alternative definition of RPO as a pushout in a slice-category, we define a biRPO to be a bipushout in a pseudo-slice-category. The reader may wish to unravel the definition and check that GRPOs (cf. Definition 4.3) are biRPOs in a G-category.

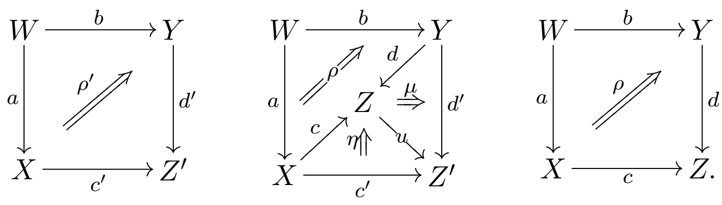
# B Proof of Lemma 4.8

**Proof.** Suppose that $\mathbf{R} = \langle R, e, f, g, \beta, \gamma, \delta \rangle$ is a candidate for *(ii)*. Then then tuple $\mathbf{R}' = \left\langle R, e, f, g, \beta \bullet e\epsilon^{-1}, \gamma, \delta \right\rangle$ is a candidate for *(i)* and we obtain the mediating morphism $\langle u, \phi, \psi, \tau \rangle$ between $\langle Z, c, d, \mathrm{id}_Z, \rho, 1_c, 1_d \rangle$ and $\mathbf{R}'$. It is straightforward to check that this is also a mediating morphism between $\langle Z, c, d, \mathrm{id}_Z, \rho \bullet c\epsilon, 1_c, 1_d \rangle$ and $\mathbf{R}$ and that the universal property follows from the universal property of *(i)*.

If $\langle R, e, f, g, \beta, \gamma, \delta \rangle$ is a candidate for *(iii)*, then $\left\langle R, e, f, g, \beta, \gamma, \epsilon'^{-1} \bullet \delta \right\rangle$ is a candidate for *(i)*. Hence there is a mediating morphism $\langle u, \phi, \psi, \tau \rangle$ and it is easy to check that $\left\langle u, \phi, \psi \bullet u\epsilon'^{-1}, \tau \right\rangle$ is a mediating morphism for the region. It is clear that the universal property also follows. □

# C Proof of Theorem 4.10

**Lemma C.1 (GIPOs from GRPOs)** *If $\langle Z, c, d, u, \rho, \eta, \mu \rangle$ is a GRPO for (i) below, as illustrated in (ii), then (iii) is a GIPO.*




*(i)* *(ii)* *(iii)*

***Proof.*** Suppose that $\langle R, e, f, g, \beta, \gamma, \delta\rangle$ is a candidate for *(iii)*. Then it is easy to verify that $\langle R, e, f, ug, \beta, u\gamma \bullet \eta, \mu \bullet u\delta\rangle$ is a candidate for *(i)*.

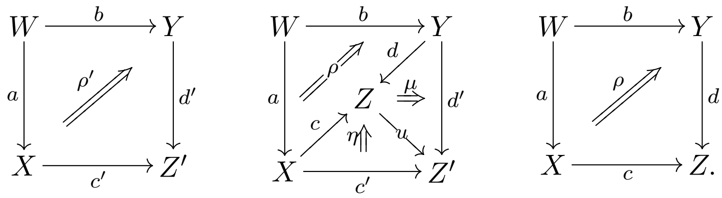
Thus there exists an arrow $h\colon Z \to R$ and isomorphisms $\varphi\colon e \Rightarrow hc$, $\psi\colon hd \Rightarrow f$ and $\tau\colon ugh \Rightarrow u$ satisfying $\tau c \bullet ug\varphi \bullet u\gamma \bullet \eta = \eta$, $\mu \bullet u\delta \bullet ug\psi \bullet \tau^{-1}d = \mu$ and $\psi b \bullet h\rho \bullet \varphi a = \beta$ (†).

It follows that $\langle id_Z, 1_c, 1_d, 1_u\rangle$ and $\langle gh, g\varphi \bullet \gamma, \delta \bullet g\psi, \tau\rangle$ are both mediating morphisms from $\langle Z, c, d, u, \rho, \eta, \mu\rangle$ to $\langle Z, c, d, u, \rho, \eta, \mu\rangle$. Therefore there exists a unique 2-cell $\xi\colon gh \Rightarrow \mathrm{id}_Z$ such that $\xi c \bullet g\varphi \bullet \delta = 1_c$ (‡), $\delta \bullet g\psi \bullet \xi^{-1}d = 1_d$ (♠) and $u\xi = \tau$.

Equations (†), (‡) and (♠) ensure that $\langle h, \varphi, \psi, \xi\rangle$ is a mediating morphism from $\langle Z, c, d, \mathrm{id}, \rho, 1_c, 1_d\rangle$ to $\langle R, e, f, g, \beta, \delta, \gamma\rangle$ as candidates for *(iii)*.

Let $\langle h', \varphi', \psi', \xi'\rangle$ be another such mediating morphism. Then it is easy to verify that $\langle h', \varphi', \psi', u\xi'\rangle$ constitutes another mediating morphism from $\langle Z, c, d, u, \rho, \eta, \mu\rangle$ to $\langle R, e, f, ug, \beta, u\gamma \bullet \eta, \mu \bullet u\delta\rangle$. Thus there exists an unique $\lambda\colon h \Rightarrow h'$ which satisfies $\lambda c \bullet \varphi = \varphi'$, $\psi \bullet \lambda^{-1}d = \psi'$ and $u\xi' \bullet ug\lambda = \tau (= u\xi)$. It remains to check that $\xi' \bullet g\lambda = \xi$, and this follows from the uniqueness of $\xi$. □

**Lemma C.2 (GRPOs from GIPOs)** *If (iii) below is a* G*IPO, (i) has a* G*RPO, and* $\langle Z, c, d, u, \rho, \eta, \mu\rangle$ *is a candidate for it as shown in (ii), then* $\langle Z, c, d, u, \rho, \eta, \mu\rangle$ *is a* G*RPO for (i).*




*(i)* *(ii)* *(iii)*

***Proof.*** $\langle R, e, f, g, \beta, \gamma, \delta\rangle$ be an RPO for *(i)*. Using its defining property, there exists a morphism $v\colon R \to Z$ and isomorphisms $\varphi\colon c \Rightarrow ve$, $\psi\colon vf \Rightarrow d$ and $\tau\colon uv \Rightarrow g$ which satisfy $\tau e \bullet u\varphi \bullet \eta = \gamma$ $(\star)$, $\mu \bullet u\psi \bullet \tau^{-1}f = \delta$ and $\psi b \bullet v\beta \bullet \varphi a = \rho$ (†). The last equation asserts that $\langle R, e, f, v, \beta, \varphi, \psi\rangle$ is a candidate for the square on the right, thus there exists an arrow $w\colon Z \to R$ and isomorphisms $\varphi'\colon e \Rightarrow wc$, $\psi'\colon wd \Rightarrow f$ and $\tau'\colon vw \Rightarrow \mathrm{id}_Z$ such that $\tau' c \bullet v\varphi' \bullet \varphi = 1_c$ $(\star\star)$, $\psi \bullet v\psi' \bullet {\tau'}^{-1}d = 1_d$ and $\psi' b \bullet w\rho \bullet \varphi' a = \beta$ (‡).

We claim that $\langle wv, w\varphi \bullet \varphi', \psi' \bullet w\psi, \tau \bullet u\tau' v \bullet \tau^{-1}wv\rangle$ and $\langle \mathrm{id}_R, 1_e, 1_f, 1_g\rangle$ are mediating morphisms from $\langle R, e, f, g, \beta, \gamma, \delta\rangle$ to $\langle R, e, f, g, \beta, \gamma, \delta\rangle$. Firstly, (†) and (‡) together imply that $(\psi' \bullet w\psi)b \bullet wv\beta \bullet (w\varphi \bullet \varphi')a = \beta$. It remains to show that

$$(\tau \bullet u\tau' v \bullet \tau^{-1}wv)e \bullet g(w\varphi \bullet \varphi') \bullet \gamma = \gamma$$

and

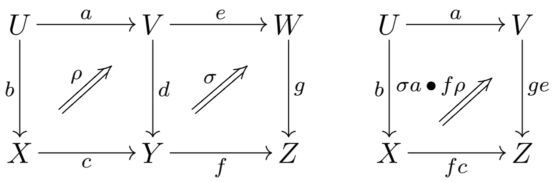
$$\delta \bullet g(\psi' \bullet w\psi) \bullet (\tau \bullet u\tau' v \bullet \tau^{-1}wv)^{-1}f = \delta.$$

We shall show the first holds, the second is similar. Indeed,

$$\begin{aligned}
&(\tau \bullet u\tau' v \bullet \tau^{-1} wv)e \bullet g(w\varphi \bullet \varphi') \bullet \gamma = && \text{(pasting)}\\
&\tau e \bullet u\varphi \bullet u\tau' c \bullet uv\varphi' \bullet \tau^{-1} e \bullet \gamma = && (\star)\\
&\tau e \bullet u\varphi \bullet u\tau' c \bullet uv\varphi' \bullet u\varphi \bullet \eta = && \\
&\tau e \bullet u\varphi \bullet u(\tau' c \bullet v\varphi') \bullet u\varphi \bullet \eta = && (\star\star)\\
&\tau e \bullet u\varphi \bullet \eta = \gamma. && (\star)
\end{aligned}$$

Therefore, there exists a unique $\xi\colon wv \Rightarrow \mathrm{id}_R$ which makes the two mediating morphisms compatible. Since GRPOs are defined up to an equivalence, this completes the proof. □

**Lemma C.3** Suppose that diagram *(ii)* below has a GRPO.



*(i)* *(ii)*

(i) If both squares in *(i)* are GIPOs then the rectangle of *(i)* is a GIPO

(ii) If the left square and the rectangle of *(i)* are GIPOs then so is the right square.

**Proof. (1).** By Lemma C.2, $\langle Y, c, d, f, \rho, 1_{fc}, \sigma\rangle$ is a GRPO for *(ii)*. Suppose that $\langle R, u, v, w, \beta, \gamma, \delta\rangle$ is a candidate for the rectangle of *(i)*, that is $\delta ea \bullet w\beta \bullet \gamma b = \sigma a \bullet f\rho$.

Thus $\langle R, u, ve, \beta, \gamma, \delta e\rangle$ is a candidate for *(ii)* and so there exists an arrow $m\colon Y \to R$ and two-cells $\varphi\colon u \Rightarrow mc$, $\psi\colon md \Rightarrow ve$ and $\tau\colon wm \Rightarrow f$ satisfying the usual compatibility requirements.

In particular, $\delta e \bullet w\psi \bullet \tau^{-1} d = \sigma$, and so $\langle R, m, v, w, \psi, \tau^{-1}, \delta\rangle$ is a candidate for the right square of *(i)*. Thus there exists an arrow $n\colon Z \to R$ and two-cells $\varphi'\colon m \Rightarrow nf$, $\psi'\colon ng \Rightarrow v$ and $\tau'\colon wn \Rightarrow \mathrm{id}_Z$. The reader should verify that $\langle n, \varphi' c \bullet \varphi, \psi', \tau'\rangle$ is a mediating morphism from $\langle Z, fc, g, \mathrm{id}_Z, \sigma a \bullet f\rho, 1_{fc}, 1_g\rangle$ to $\langle R, u, v, w, \beta, \gamma, \delta\rangle$.

Let $\langle n', \varphi'', \psi'', \tau''\rangle$ be another such mediating morphism. Then it follows that $\langle n' f, \varphi'', \psi'' e \bullet n'\sigma, \tau'' f\rangle$ is a mediating morphism from candidate $\langle Y, c, d, f, \rho, 1_{fc}, \sigma\rangle$ to $\langle R, u, ve, \beta, \gamma, \delta e\rangle$. Thus there exists a unique $\xi\colon m \Rightarrow n' f$ which makes it compatible with $\langle m, \varphi, \psi, \tau\rangle$, in particular, $\xi c \bullet \varphi = \varphi''$. Now $\langle n', \xi, \psi'', \tau''\rangle$ is a mediating morphism between $\langle Z, f, g, \mathrm{id}_Z, \sigma, 1_f, 1_g\rangle$ and $\langle R, m, v, w, \psi, \tau^{-1}, \delta\rangle$. Hence there exists a unique $\xi'\colon n \Rightarrow n'$ which makes the mediating morphism compatible with $\langle n, \varphi', \psi', \tau'\rangle$. It is easy to check that $\xi'$ makes $\langle n, \varphi' c \bullet \varphi, \psi', \tau'\rangle$ compatible with $\langle n', \varphi'', \psi'', \tau''\rangle$ also. If there is another such $\xi''$ then by uniqueness of $\xi$ it also makes $\langle n, \varphi', \psi', \tau'\rangle$ and $\langle n', \xi, \psi'', \tau''\rangle$ compatible, hence $\xi''$ must equal $\xi'$.

**(2).** Suppose that $\langle R, u, v, w, \beta, \delta, \gamma\rangle$ is a candidate for the right square of *(i)*. Then

$$\langle R,\ uc,\ v,\ w,\ \beta a \bullet u\rho,\ \gamma c,\ \delta\rangle$$

is a candidate for the rectangle and so there exists an arrow $m\colon Z \to R$ and two-cells $\varphi\colon uc \Rightarrow mfc$, $\psi\colon mg \Rightarrow v$ and $\tau\colon wm \Rightarrow \mathrm{id}_Z$ satisfying the three compatibility equations.

Recall that by Lemma C.2, candidate $\langle Y, c, d, f, \rho, 1_{fc}, \sigma\rangle$ is a GRPO for *(ii)*. Now $\left\langle u, 1_{uc}, \beta, \gamma^{-1}\right\rangle$ and $\langle mf, \varphi, \psi e \bullet m\sigma, \tau f\rangle$ are mediating morphisms between

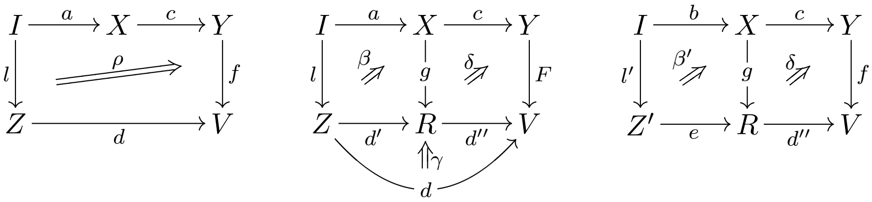
$$\langle Y, c, d, f, \rho, 1_{fc}, \sigma\rangle \quad \text{and} \quad \langle R, uc, ve, w, \beta a \bullet u\rho, \gamma, \delta\rangle\,.$$

Thus there exists a unique two-cell $\xi\colon u \Rightarrow mf$ making the two mediating morphisms compatible. In particular , $\xi c = \varphi$ which implies that $\langle m, \xi, \psi, \tau\rangle$ is a mediating morphism from $\langle Z, f, g, \mathrm{id}_Z, \sigma, 1_f, 1_g\rangle$ to $\langle R, u, v, \beta, \gamma, \delta\rangle$ in the right square.

If $\langle m', \varphi', \psi', \tau'\rangle$ is another such mediating morphism then $\langle m', \varphi' c, \psi', \tau'\rangle$ is a mediating morphism for the rectangle. Hence there exists a unique $\xi'\colon m \Rightarrow m'$ which makes this mediating morphism compatible with $\langle m, \varphi, \psi, \tau\rangle$. The universal property of the left square implies that $\xi'$ also makes $\langle m, \xi, \psi, \tau\rangle$ compatible with $\langle m', \varphi', \psi', \tau'\rangle$. Uniqueness follows from the universal property of the rectangle. $\square$

## Proof of Theorem 4.10

**Proof.** It suffices to show that $S = \{\, (ca, cb) \mid a \sim b \,\}$ is a bisimulation. Suppose that $a \sim b$ and $ca \xrightarrow{[f]}\!\!\rhd\, a'$. Then




(*i*) (*ii*) (*iii*)

there exists $\langle l, r\rangle \in \mathcal{R}$, $d\colon Z \to V$ and $\rho\colon dl \Rightarrow fca$ such that (i) is a GIPO and $a' = dr$. Since $\mathbb{C}$ is has redex-GRPOs, there exists $\langle R, d', g, d'', \beta, \gamma, \delta\rangle$ as shown in (ii) which is a GRPO. By Lemma C.1, the left square in (ii) is a GIPO. Thus $a \xrightarrow{[g]}\!\!\rhd\, d'r$ and so $b \xrightarrow{[g]}\!\!\rhd\, b'$ where $b' \sim d'r$. By definition, there is a pair $\langle l', r'\rangle \in \mathcal{R}$, an arrow $e\colon Z' \to R$ and a two-cell $\beta'\colon el' \Rightarrow gb$ so that the left square of (iii) is a GIPO and $b' = er'$.

Now Lemma 4.8 implies that the composite of the two squares in (ii) is a GIPO and therefore, by part 2 of Lemma C.3 the right square is a GIPO. Since we have deduced that both the squares in (iii) are GIPOs, part 1 of Lemma C.3 ensures that the entire region is a GIPO and that $cb \xrightarrow{[f]}\!\!\rhd\, d''er'$. Since $d'r \sim er'$, we conclude that $(d''d'r, d''er') \in S$. $\square$