Toposes are adhesive

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Abstract. Adhesive categories have recently been proposed as a categorical foundation for facets of the theory of graph transformation, and have also been used to study techniques from process algebra for reasoning about concurrency. Here we continue our study of adhesive categories by showing that toposes are adhesive. The proof relies on exploiting the relationship between adhesive categories, Brown and Janelidze's work on generalised van Kampen theorems as well as Grothendieck's theory of descent.

Introduction

Adhesive categories [11,12] and their generalisations, quasiadhesive categories [11] and adhesive HLR categories [6], have recently begun to be used as a natural and relatively simple general foundation for aspects of the theory of graph transformation, following on from previous work in this direction [5]. By covering several "graph-like" categories, they serve as a useful framework in which to prove structural properties. They have also served as a bridge allowing the introduction of techniques from process algebra to the field of graph transformation [7, 13].

From a categorical point of view, the work follows in the footsteps of distributive and extensive categories [4] in the sense that they study a particular relationship between certain finite limits and finite colimits. Indeed, whereas distributive categories are concerned with the distributivity of products over coproducts and extensive categories with the relationship between coproducts and pullbacks, the various flavours of adhesive categories consider the relationship between certain pushouts and pullbacks.

Adhesive categories are defined to be the categories with pullbacks where pushouts along monomorphisms are van Kampen [11] and as a consequence such pushouts can be considered as being well-behaved with respect to pullbacks. As we shall explain, related work includes Grothendieck's theory of descent (see [9] for an overview) and generalised approaches to the van Kampen theorem [1].

As shown in [11], adhesive categories are closed under several categorical constructions, thus if $\mathbf{C}$ and $\mathbf{D}$ are adhesive then so is their product $\mathbf{C} \times \mathbf{D}$, choosing any object $C \in \mathbf{C}$, the slice category $\mathbf{C}/C$ and coslice category $C/\mathbf{C}$ are adhesive and, for any category $\mathbf{X}$, the functor category $[\mathbf{X}, \mathbf{C}]$ is adhesive. It is

⋆ Research partially supported by EPSRC grant EP/D066565/1. The first author gratefully acknowledges the support of the Australian Research Council.

also known that the category of sets and functions **Set** is adhesive, in particular this means that any presheaf topos $[\mathbf{X}, \mathbf{Set}]$ is adhesive. These constructions are useful because adhesive categories satisfy many of the so-called HLR-axioms, and as a consequence, several important theorems about the rewriting theory of double-pushout transformations can be proved at the level of adhesive categories. Indeed, it is perhaps surprising that so many of the axioms, which were not known previously to be related, hold automatically in any adhesive category. One of the original contributions of this paper is a proof that adhesive categories satisfy the *special pullback-pushout property*, one of the aforementioned axioms.

The central part of the paper is devoted to studying the relationship between toposes and adhesive categories. Topos theory has many different facets and applications within mathematics and computer science. One of the interesting properties of toposes is that they have finite limits and colimits, and these behave somewhat as they do in **Set**. Indeed, while toposes enjoy much more structure than adhesive categories, adhesive categories themselves have certain finite colimits (pushouts along monomorphisms) and limits (pullbacks) which are well-behaved with respect to each other. The question of whether toposes are adhesive is thus a very natural one.

As we have shown in [11, 12], there are adhesive categories which are not toposes. Here we prove that the converse is not true – indeed, the main contribution of the paper is Theorem 26, the conclusion of which states that toposes are adhesive. From a computer science perspective, this means that the theory developed for adhesive categories can be applied to any topos, not just a presheaf. This is a significant development, since topos theory is a well-established mathematical discipline with many important results and wide-reaching applications.

One interesting example of a category which is a topos and not a presheaf is the *Schanuel topos*. It has been used (see [8], for example) to model languages with name binding, such as the Pi-calculus. Our theorem thus allows us to apply the rewriting theory developed for adhesive categories to such a setting. While we do not study this example in detail within the present paper, we plan to study such systems as part of future work. The Schanuel topos actually arises as a full subcategory of a presheaf category with objects the (atomic) *sheaves*. It is a general fact that any such category of sheaves is a topos.

The proof of our theorem relies on exploiting the connections between adhesive categories (or more generally, van Kampen squares), Brown and Janelidze's generalised van Kampen theorems and Grothedieck's theory of descent. Indeed, in order to prove that toposes are adhesive, we must show that pushouts along monomorphisms are van Kampen. To do so, we split such a pushout into two: one pushout with all morphisms mono, and one with two monomorphisms and two epimorphisms. The former is also a pullback in any topos, and a theorem of Brown and Janelidze's (here recalled as Theorem 23) guarantees that it satisfies the van Kampen theorem – which implies that the original pushout is a van Kampen square. Here we also prove that pushouts of the latter kind are van Kampen in a topos, it is the most difficult and technical result of the paper.

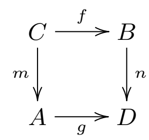


Fig. 1. Pushout diagram.

This completes the proof because, as we show in Lemma 2, van Kampen squares compose in categories with pullbacks and pushouts.

Structure of the paper. In Section 1 we recall two equivalent ways of defining van Kampen squares and show that van Kampen squares compose in categories with pushouts and pullbacks. We recall the definition of adhesive categories and prove that adhesive categories enjoy the special pullback-pushout property. Section 2 recalls the fragments of descent theory and topos theory necessary for our main result. In Section 3, we recall the theorem of Brown and Janelidze, which forms one of the ingredients of the proof of our main Theorem 26. Theorem 25 is the other main ingredient, and its proof relies on the background introduced in Section 2. We conclude in Section 4 with several directions for future work. The paper is relatively self-contained, although we omit the proofs of well-known results and instead provide references to standard sources.

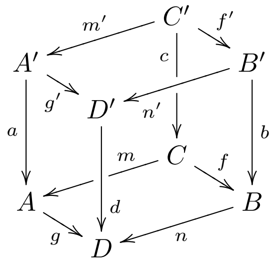
1 Van Kampen squares and adhesive categories

Adhesive and quasiadhesive categories are defined using certain pushout diagrams which are called van Kampen squares. We refer the reader to [11, 12] for an introduction to, the enumeration of the basic properties of, and the applications of adhesive and quasiadhesive categories. Here we shall concentrate on the definitions of van Kampen squares and derive the properties which will be needed for the proof of our main result. We shall also prove that adhesive categories satisfy the so-called special pullback-pushout property, which is one of the many HLR axioms, the majority of which have already been shown in [11, 12] to hold in any adhesive category.

We shall need both the axiomatic and the "equivalence of categories" versions of the definitions of van Kampen squares [12]. In particular, using the former, we shall show that van Kampen squares compose in categories with pushouts and pullbacks, and that adhesive categories satisfy the special pullback-pushout property. The fact that van Kampen squares compose, together with the latter, equivalent, way of defining van Kampen squares will be used in the proof of our main Theorem 26. The latter definition of van Kampen squares also makes it possible to establish a relationship between van Kampen squares and Brown and Janelidze's generalised van Kampen theorems, as we shall explain in §3.

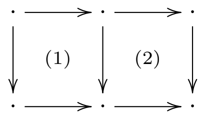
Definition 1. A van Kampen square is a pushout diagram as in Fig 1 which satisfies the following condition:

– for any commutative cube, as illustrated, of which Fig 1 forms the bottom face and the back faces are pullbacks: the front faces are pullbacks iff the top face is a pushout.



The following lemma shows that, in categories with pushouts and pullbacks, van Kampen squares paste together to give van Kampen squares.

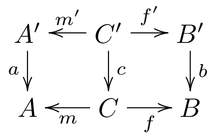
Lemma 2. *Consider the illustrated commutative diagram in a category with pushouts and pullbacks. If (1) and (2) are van Kampen then so is (1)+(2).*



Proof. Straightforward; in order to show that the combined pushout is stable under pullback it suffices to break up a cube into two cubes, using the existence of pullbacks. Conversely, a cube with its top face a pushout, can be split into two using the existence of pullbacks and pushouts. □

We shall now recall an equivalent definition of van Kampen squares which will be useful for the purposes of this paper. The reader is referred to [12] for the proof that the definitions are equivalent. The alternative definition is stated by saying that a certain functor, induced by the diagram in Fig 1, is required to be an equivalence of categories. We begin by defining the codomain category of the functor.

Definition 3. Let $\mathbf{C}/A \times_{\mathbf{C}/C} \mathbf{C}/B$ denote the category with objects commutative diagrams of pullbacks, as illustrated, and arrows the obvious morphisms between such diagrams.



For a morphism $u\colon U \to V$ we shall write $u^*\colon \mathbf{C}/V \to \mathbf{C}/U$ for the functor given by pulling back along u. Referring to Fig 1, the functors n^* and g^* induce a functor

$$\mathsf{Pb}\colon \mathbf{C}/D \to \mathbf{C}/A \times_{\mathbf{C}/C} \mathbf{C}/B.$$

Using the functor Pb, we can define the property of square (1) being van Kampen as follows:

Definition 4. The pushout diagram of Fig 1 is said to be van Kampen whenever one of the following equivalent conditions holds:

(i) Pb is an equivalence of categories;

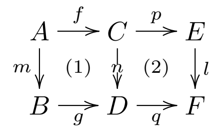
(ii) the pushout is stable under pullback, and the functor Pb is essentially surjective on objects.[3]

Definition 5 (Adhesive categories). A category with pullbacks and pushouts along monomorphisms is said to be adhesive if any pushout square as in Fig 1, in which m is a monomorphism, is van Kampen.

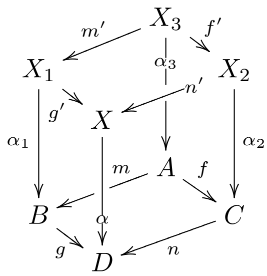
Examples of adhesive categories include **Set** (see [11]) and the category of graphs **Graph**. The fact that the latter is adhesive follows from the fact that **Set** is adhesive and the fact that the functor category $[\mathbf{X}, \mathbf{C}]$ is adhesive whenever $\mathbf{C}$ is. Thus, in particular, any presheaf topos is adhesive. In §3 we shall show that any topos is adhesive, thus providing several new examples of adhesive categories – for instance, the Schanuel topos [8].

Adhesive categories have found an application as a foundation for parts of the theory of graph transformation. Indeed, it has been shown in [11, 12] that adhesive and quasiadhesive categories satisfy many of the previously proposed HLR-axioms [5]. Here we shall extend this thesis by showing that another of the aforementioned axioms holds in adhesive categories, the so called special pullback-pushout property. Actually, we are able to prove a more general property by requiring less assumptions about the arrows in the diagram below.

Lemma 6 (Special pullback-pushout property). *Suppose that the illustrated commutative diagram in an adhesive category has m, n and l mono. Suppose that (1) is a pushout and (1)+(2) is a pullback, then (2) is a pullback.*



Proof. Suppose we have an object X and morphisms $\alpha\colon X \to D$ and $\beta\colon X \to E$ such that $q\alpha = l\beta$. We shall show that there exists $k\colon X \to C$ such that $nk = \alpha$ and $pk = \beta$. Notice that it suffices to show the existence of such a morphism, uniqueness follows since n is mono.



Construct the illustrated cube by taking pullbacks. Now $qg\alpha_1 = q\alpha g' = l\beta g'$, and we use the fact that (1)+(2) is a pullback to derive the existence of a unique morphism $h\colon X_1 \to A$ such that $mh = \alpha_1$ and $pfh = \beta g'$.

Also note that $m\alpha_3 = \alpha_1 m' = mhm'$, and using the fact that m is mono, $\alpha_3 = hm'$ (†). Also, $lp\alpha_2 = qn\alpha_2 = q\alpha n' = l\beta n'$. Since l is mono, we have that $p\alpha_2 = \beta n'$ (‡).

We shall use the fact that the top face of the cube is a pushout to derive the existence of the required morphism. Indeed, we have $\alpha_2 f' = f\alpha_3 = fhm'$ where we used (†) to derive the last equality. Thus we get a unique $k\colon X \to C$ such that $kg' = fh$ and $kn' = \alpha_2$.

[3] A functor $F\colon \mathbf{C} \to \mathbf{D}$ is said to be *essentially surjective on objects* when, for every object $D \in \mathbf{D}$, there exists an object $C \in \mathbf{C}$ such that $FC \cong D$.

It remains to show that k satisfies the necessary properties, that is, $nk = \alpha$ and $pk = \beta$. Indeed, we have $nkg' = nfh = gmh = g\alpha_1 = \alpha g'$ and $nkn' = n\alpha_2 = \alpha n'$. Using the fact that the top face of the cube is a pushout, and in particular, the uniqueness of the mediating morphism, we have $nk = \alpha$.

Similarly, $pkg' = pfh = \beta g'$ and $pkn' = p\alpha_2 = \beta n'$, where we used (‡) to derive the last equality. This implies that $pk = \beta$ and we are finished. □

2 Toposes and descent

In order to prove that toposes are adhesive, we shall first recall the necessary background in this section: basic aspects of descent theory as well as the definition and several well-known properties of toposes.

We start in §2.1 by recalling a little of Grothendieck's theory of descent. We refer the reader to [9], for example, for a more detailed account. Historically, the theory arose from algebraic geometry and has had an impact on many disciplines within mathematics and computer science, including algebraic topology, logic and type theory. Descent theory, as we shall show, is also closely related to van Kampen squares.

In §2.2 we recall some elementary facts about toposes. See [10] for an overview of topos theory. Toposes have been widely used by mathematicians and computer scientists interested in logic, topology, geometry or category theory.

We relate these two topics by recalling that epimorphisms in toposes are effective for descent. This, in fact, is a consequence of a more general fact that regular epimorphisms in locally-cartesian closed categories are effective descent morphisms.

2.1 Descent

Recall that a morphism $p\colon X \to D$ is said to be a regular epimorphism if it is the coequaliser of some morphisms $w_1, w_2\colon W \to X$. We recall below a well-known lemma which relates regular epimorphisms and pushout diagrams.

Lemma 7. *If $p\colon X \to D$ has a kernel pair $p_1, p_2\colon P \to X$ (the diagram is a pullback square) then p is a regular epimorphism iff the pullback is also a pushout.*

$$\begin{array}{ccc} P & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & D \end{array}$$

Proof. (⇐) If the diagram is a pushout square then it follows immediately that p is a coequaliser of p_1 and p_2;
(⇒) If p is regular epi then it is the coequaliser of $w_1, w_2\colon W \to X$. Since the diagram above is a pullback, there exists a unique morphism $w\colon W \to P$ such that $p_1 w = w_1$ and $p_2 w = w_2$. From this it follows that p is the coequaliser of p_1 and p_2. Again using the fact that the square is a pullback, there exists an arrow $h\colon X \to P$ such that $p_1 h = \mathrm{id}_X$ and $p_2 h = \mathrm{id}_X$. It follows that for any $\alpha\colon X \to Y$ and $\beta\colon X \to Y$ if $\alpha p_1 = \beta p_2$ then $\alpha = \beta$. The universal property of pushouts thus follows from the universal property of coequalisers. □

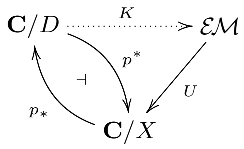


Fig. 2. The adjunction $p_* \dashv p^*$ and functor to the Eilenberg-Moore category $\mathcal{EM}$, with the obvious forgetful functor U.

The conclusion of the following lemma ensures that the evident induced morphism to the vertex of any pushout diagram is a regular epimorphism.

Lemma 8. *Given a pushout diagram as in Fig 1 in a category with coproducts,* $[g, n]\colon A + B \to D$ *is a regular epimorphism.*

Proof. It is the coequaliser of the pair $i_1 m, i_2 f\colon C \to A + B$. □

We shall now recall some basic facts about descent. Let $\mathbf{C}$ be a category with pullbacks and $p\colon X \to D$ a morphism in $\mathbf{C}$. The pullback functor $p^*\colon \mathbf{C}/D \to \mathbf{C}/X$ always has a left adjoint, given by composition with p. Recall that the right adjoint $p^*\colon \mathbf{C}/D \to \mathbf{C}/X$ is said to be *monadic* if the unique comparison functor K from $\mathbf{C}/D$ to the Eilenberg-Moore category $\mathcal{EM}$ generated by the monad arising from the adjunction is an equivalence of categories (see Fig 2).

The following definition states when a morphism is said to be *effective for descent*. The intuitive idea is that, given an effective descent morphism $p\colon X \to D$, one can reason about the structure of a category $\mathbf{C}/D$, which may be difficult, by reasoning about certain algebras over $\mathbf{C}/X$ – thus in a sense "descending" along p. Indeed, referring to the diagram of Fig 2, to say that p is effective for descent is to say that the comparison functor $K\colon \mathbf{C}/D \to \mathcal{EM}$, where $\mathcal{EM}$ is the Eilenberg-Moore category (category of algebras) of the monad with endofunctor $p^* p_*\colon \mathbf{C}/X \to \mathbf{C}/X$, is an equivalence of categories; ie p^* is monadic.

Definition 9. If p^* is monadic, we shall say either that p is *effective for descent* or that it is an *effective descent* morphism.

In the particular case of the monad $p_* \dashv p^*$, the Eilenberg-Moore category can be characterised as a category of certain pullback diagrams in $\mathbf{C}$. This characterisation will prove useful in the proof of our main result.

Lemma 10. *The Eilenberg-Moore category* $\mathcal{EM}$ *of the monad induced by the adjunction* $p_* \dashv p^*$ *is the category whose:*

- *objects are diagrams of pullbacks, as illustrated in the left diagram of Fig 3, where* $p_1, p_2\colon P \to X$ *is the kernel pair of* p*;*
- *arrows are pairs* α, β *which combine into a commutative diagram, as illustrated in the other diagram of Fig 3.*

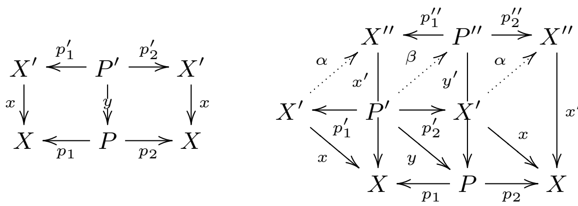


Fig. 3. Objects and morphisms of $\mathcal{EM}$.

The comparison functor $K\colon \mathbf{C}/D \to \mathcal{EM}$ *(see Fig 2) takes an object* $d\colon D' \to D$ *of* $\mathbf{C}/D$ *to the rear faces of the cube of pullbacks formed from* d *and the pullback diagram of Lemma 7.*

Proof. See [9, §3.4]. Note that the category discussed there is the category of descent data: its objects are pairs $\langle x\colon X' \to X,\ \xi\colon p_1^*x \to p_2^*x\rangle$ where ξ is an iso in $\mathbf{C}/P$. This category is clearly isomorphic to the category described above. □

Remark 11. The category $\mathcal{EM}$ is *not* the same as $\mathbf{C}/X \times_{\mathbf{C}/P} \mathbf{C}/X$ of Definition 3. Indeed, if p is a regular epimorphism and we start with the pushout diagram of Lemma 7, then it makes sense to compare the two categories. While the objects of both are pairs of pullback diagrams, an object of the latter category can potentially involve two maps $X' \to X$ rather than one.

The next lemma relates the effective descent morphisms and regular epimorphisms. The two classes coincide in any locally cartesian closed category – a category $\mathbf{C}$ in which every slice $\mathbf{C}/C$ is cartesian closed.

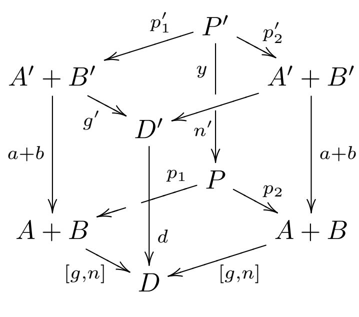
Lemma 12. *In a locally cartesian closed category, a morphism is effective for descent (cf Definition 9) iff it is a regular epimorphism.*

Proof. See [9], for example. □

We have the following useful fact as a consequence of Lemmas 12 and 8. It states that, in a category with coproducts, the evident morphism induced by an arbitrary pushout is effective for descent.

Corollary 13. *Given a pushout diagram of Fig 1 in a locally cartesian closed category with coproducts, the induced arrow* $[g, n]\colon A + B \to D$ *is an effective descent morphism.* □

Now letting $p = [g, n]$ and using Corollary 13, Lemma 10, the comparison functor $K\colon \mathbf{C}/D \to \mathcal{EM}$ which takes an object $d\colon D' \to D$ to the back faces of the cube of pullbacks, as illustrated, is an equivalence of categories. This fact, and in particular the fact that K is essentially surjective on objects, will form an important part of the proof of Theorem 25, the hardest step of the proof of our main Theorem 26.



2.2 Toposes

Here we list the properties of toposes which we shall use to prove our main theorem. We refer the reader to [10] for a more thorough account of topos theory. We give a standard definition of toposes below; note that the actual statement of the definition is not important for the purposes of this paper, instead we shall list the precise properties of toposes we require in the remainder of this section.

Definition 14. A topos is a category $\mathbf{C}$ which:

(i) is cartesian closed and has equalisers (and consequently, all finite limits);
(ii) has a subobject classifier.

It follows from the axioms above that toposes have finite colimits [10, A2.2.9] and are locally cartesian closed [10, A2.3.4]. In particular, the latter implies that:

Proposition 15. *The pullback functors $u^* \colon \mathbf{C}/V \to \mathbf{C}/U$ have right adjoints, and so preserve all colimits.* □

The proof of Theorem 25, the hardest step in the proof of our main result, relies on the fact that toposes are *extensive* [4]. Extensive categories can be said to have well-behaved coproducts, in a similar sense to how pushouts along monomorphisms can be said to be well-behaved in adhesive categories. Here we give the (axiomatic) definition and a well-known characterisation.

Definition 16 (Extensive categories). A category $\mathbf{C}$ is said to be *extensive* if it has finite coproducts, pullbacks along coproduct injections, and satisfies the following property:

– given a commutative diagram, as illustrated, the top row is a coproduct diagram iff the two squares are pullbacks.

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow[i_1]{} & A+B & \xleftarrow[i_2]{} & B \end{array}$$

The following result states two properties of coproducts in extensive categories which, if they hold in arbitrary categories, are enough to show extensivity. Interestingly, it is unknown whether a similar characterisation can be given for adhesive categories.

Proposition 17. *A category with finite coproducts and pullbacks along coproduct injections is extensive iff (1) it has coproducts which are stable under pullback and (2) pulling back one coproduct injection along the other results in the initial object.*[4]

Proof. See [4, Proposition 2.14]. □

[4] Coproducts are said to be *disjoint* if they satisfy property (2) and coproduct injections are monomorphisms. When coproducts are stable under pullback, the fact that injections are monomorphisms is derivable from (2) (cf [4, Lemma 2.13]).

We shall now recall some other well-known properties of toposes, one of which is that toposes are extensive:

Lemma 18. *If* $\mathbf{C}$ *is a topos then:*

(i) *epimorphisms in* $\mathbf{C}$ *are regular and are stable under pullback;*
(ii) *monomorphisms in* $\mathbf{C}$ *are regular and are stable under pushout;*
(iii) *pushouts along monomorphisms in* $\mathbf{C}$ *are pullbacks;*
(iv) $\mathbf{C}$ *is extensive;*
(v) *all arrows* $f\colon A \to B$ *in* $\mathbf{C}$ *can be factorised into an epimorphism* $e\colon A \to C$ *followed by a monomorphism* $m\colon C \to B$ *with* $me = f$*; moreover, the factorisation is unique up to isomorphism in the obvious way.*

Proof. For the first parts of (*i*) and (*ii*) see [10, A1.4.9]. The fact that epimorphisms are stable under pullback follows from Proposition 15. For the second part of (*ii*) and for (*iii*) see [10, A2.4.3]. For part (*iv*), we know from Proposition 17 that it is enough to check for stability of coproducts under pullback and disjointness. For disjointness see [10, A2.4.4], stability follows from Proposition 15. For part (*iv*) see [10, A2.3.5]. □

Using Lemma 12 and part (*i*) of Lemma 18, we obtain the following well-known result:

Corollary 19. *In a topos, the classes of epimorphisms and effective descent morphisms coincide.* □

We shall require one technical lemma, which holds in toposes. It concerns certain diagrams of pullbacks – indeed, it is well known that given a diagram (as illustrated below) where the right square is a pullback then the left square is a pullback if and only if the exterior of the diagram is a pullback. The lemma below gives a sufficient condition for the right square being a pullback when the exterior of the diagram and the left square are pullbacks.

Lemma 20. *Consider in a topos a diagram, as illustrated, with* g *an epimorphism. If the left square and the exterior are pullbacks then so is the right square.*

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xrightarrow{p} & E \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} \\ B & \underset{g}{\twoheadrightarrow} & D & \underset{q}{\rightarrow} & F \end{array}$$

Proof. This follows immediately from [2, Lemma 4.6] and Corollary 19. Indeed, as shown in [2], it suffices to require that g is an effective descent morphism. □

We conclude this section by recalling the notion of an equivalence relation in a cartesian category, a generalisation of the usual notion of equivalence relation, and recalling that equivalence relations in toposes are *effective*.

Definition 21 (Equivalence relation). Suppose that $\mathbf{C}$ is a category with finite products. By a *relation*, we mean a monomorphism $\langle a, b\rangle : R \to A \times A$. A relation is said to be an *equivalence relation* if all three of the following hold:

– it is *reflexive*: there exists a morphism $r\colon A \to R$ such that $ar = br = \mathrm{id}_A$;
– it is *symmetric*: there exists a morphism $s\colon R \to R$ such that $as = b$ and $bs = a$;
– it is *transitive*: referring to the illustrated pullback diagram, there exists a morphism $t\colon P \to R$ such that $at = ap$ and $bt = bq$.

$$\begin{array}{ccc} P & \xrightarrow{q} & R \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle a} \\ R & \xrightarrow[b]{} & A \end{array}$$

Proposition 22. *In a topos, equivalence relations are* effective*, that is, they are the kernel pairs of their coequaliser.*

Proof. See [10, A2.4.1]. □

3 Toposes are adhesive

Having recalled the necessary background theory, in this section we shall prove that toposes are adhesive (Theorem 26) which is the main technical contribution of the paper. Recall from [12] that the converse does not hold – indeed there are adhesive categories which are not toposes (for instance the category of pointed sets $\mathbf{Set}_*$).

The proof itself relies on the fact that, in a topos, a pushout along a monomorphism can be broken up into two pushouts – (1) one with all arrows monomorphisms and (2) one with two monomorphisms and two epimorphisms.

Using the fact that van Kampen squares compose (cf Lemma 2), it suffices to show that, in toposes, pushouts of kinds (1) and (2) are van Kampen squares. The fact that pushouts of kind (2) are van Kampen (Theorem 25) is the most difficult and technical part of our proof. The fact that pushouts of kind (1) are van Kampen squares follows immediately from a well-known theorem of Brown and Janelidze [1]:

Theorem 23 (Brown and Janelidze). *Suppose that* $\mathbf{C}$ *is an extensive category with finite limits. Given a pullback diagram, as illustrated, with all morphisms mono, the induced (cf paragraph following Definition 3) functor* $\mathsf{Pb}\colon \mathbf{C}/D \to \mathbf{C}/A \times_{\mathbf{C}/C} \mathbf{C}/B$ *is an equivalence of categories if and only if the map* $[g, n]\colon A + B \to D$ *induced by* g *and* n *is an effective descent morphism.*

$$\begin{array}{ccc} C & \xrightarrow{f} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ A & \xrightarrow[g]{} & D \end{array}$$

Proof. See [1, Proposition 3.2]. □

As an immediate application of the above theorem, we are able to show that pushouts in toposes with all arrows monomorphisms are van Kampen.

Corollary 24. *A pushout as in Fig 1 in a topos, with all arrows monomorphic, is van Kampen.*

Proof. First note that by Lemma 18(*iii*), such a pushout is also a pullback, and by Corollary 13 we know that the arrow $[g, n]$ induced by the pushout is a (regular) epimorphism. Toposes have finite limits and are extensive (cf Lemma 18(*iv*)), thus we can apply Theorem 23 to obtain that Pb is an equivalence of categories – in other words, the pushout is van Kampen (cf Definition 4). □

The second class of pushouts we shall consider are pushouts where two of the morphisms are epimorphisms and two are monomorphisms. The following fact is the most technical part of our main result:

Theorem 25. *A pushout as in Fig 1 in a topos, with f (and so g) epimorphic and m and n monomorphic, is van Kampen.*

Proof. Using the second part of Definition 4 and the stability of pushouts under pullback (cf Proposition 15), it will suffice to show that the functor $\mathsf{Pb}\colon \mathbf{C}/D \to \mathbf{C}/A \times_{\mathbf{C}/C} \mathbf{C}/B$ induced by such a pushout is essentially surjective on objects. In other words, given a diagram as in Definition 3 with both squares pullbacks, we must find a map $d\colon D' \to D$ whose pullbacks along g and n are, respectively, a and b. By extensivity (cf Lemma 18(*iv*)), this amounts to finding d whose pullback along $p = [g, n]\colon A + B \to D$ is $a + b\colon A' + B' \to A + B$.

But, by Corollary 13, p is an effective descent map. Using the fact that $K\colon \mathbf{C}/D \to \mathcal{EM}$ is essentially surjective on objects (cf paragraph following Corollary 13), it suffices to show that the pullback of $a + b$ along p_1 coincides with its pullback along p_2,

$$\begin{array}{ccccc} A' + B' & \xleftarrow{p_1'} & P' & \xrightarrow{p_2'} & A' + B' \\ {\scriptstyle a+b}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle a+b} \\ A + B & \xleftarrow[p_1]{} & P & \xrightarrow[p_2]{} & A + B \end{array}$$

where $p_1, p_2\colon P \to A + B$ are the projections of the kernel pair of p – thus showing that the diagram is an object of the Eilenberg-Moore category $\mathcal{EM}$. By extensivity, P is given by $A_2 + C + C + B$, where $g_1, g_2\colon A_2 \to A$ is the kernel pair of $g\colon A \to D$. It follows that the projections themselves are:

$$p_1 = [g_1, m] + [f, \mathrm{id}_B]\colon A_2 + C + C + B \to A + B$$

$$p_2 = [g_2 + f,\, m + \mathrm{id}_B]\colon A_2 + C + C + B \to A + B.$$

Using extensivity once more, to show that the pullbacks of $a + b$ along p_1 and p_2 agree, it suffices to show that the pullbacks along each of the components of p_1 and p_2 agree. And since m^*a and f^*b agree, all that remains is to check that g_1^*a and g_2^*a agree. To do so, we form the pullback in diagram (i) below and then show that the squares of diagram (ii) are pullbacks.

$$\begin{array}{ccc} A_2' & \xrightarrow{\langle g_1', g_2' \rangle} & A' \times A' \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a \times a} \\ A_2 & \xrightarrow[\langle g_1, g_2 \rangle]{} & A \times A \end{array} \qquad \begin{array}{ccc} A_2' & \xrightarrow{g_i'} & A' \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a} \\ A_2 & \xrightarrow[g_i]{} & A \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{g_i' d'} & A' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow[g_i d]{} & A \end{array}$$

(i) (ii) (iii)

Let $d\colon A \to A_2$ be the unique map satisfying $g_1 d = g_2 d = \mathrm{id}_A$, and similarly let $d'\colon A' \to A'_2$ be the unique map satisfying $g'_1 d' = g'_2 d' = \mathrm{id}_{A'}$ and $a_2 d' = da$. Then the squares of diagram (iii) are clearly pullbacks.

Let $f_1, f_2\colon C_2 \to C$ be the kernel pair of f. Then there are pullback squares (iv), and so pullback squares (v). Let $m_2\colon C_2 \to A_2$ be the unique map satisfying $g_i m_2 = m f_i$ for $i = 1$ and 2. Similarly, let $m'_2\colon C'_2 \to A'_2$ be the unique map satisfying $g'_i m'_2 = m' f'_i$ for $i = 1$ and 2 as well as $a_2 m'_2 = m_2 c_2$. Thus we get the pullback squares (vi).

$$
\begin{array}{ccccc}
C'_2 & \xrightarrow{f'_i} & C' & \xrightarrow{f'} & B' \\
{\scriptstyle c_2}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\
C_2 & \xrightarrow[f_i]{} & C & \xrightarrow[f]{} & B
\end{array}
\qquad
\begin{array}{ccccc}
C'_2 & \xrightarrow{f'_i} & C' & \xrightarrow{m'} & A' \\
{\scriptstyle c_2}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} \\
C_2 & \xrightarrow[f_i]{} & C & \xrightarrow[m]{} & A
\end{array}
\qquad
\begin{array}{ccc}
C'_2 & \xrightarrow{g'_i m'_2} & A' \\
{\scriptstyle c_2}\downarrow & & \downarrow{\scriptstyle a} \\
C_2 & \xrightarrow[g_i m_2]{} & A
\end{array}
\qquad
\begin{array}{ccccc}
A' + C'_2 & \xrightarrow{[d', m'_2]} & A'_2 & \xrightarrow{g'_i} & A' \\
{\scriptstyle a + c_2}\downarrow & & {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a} \\
A + C_2 & \xrightarrow[{[d, m_2]}]{} & A_2 & \xrightarrow[g_i]{} & A
\end{array}
$$

(iv) (v) (vi) (vii)

Using extensivity and the fact that diagrams (iii) and (vi) are pullbacks, the exteriors and the left hand squares of diagram (vii) are pullbacks, so that the right hand squares will be pullbacks, and the proof complete, provided that $[d, m_2]\colon A + C_2 \to A_2$ is an epimorphism (cf Lemma 20).

To see that $[d, m_2]$ is an epimorphism, consider the map $[\Delta, \langle g_1 m_2, g_2 m_2 \rangle]\colon A + C_2 \to A \times A$ induced by the diagonal $\Delta\colon A \to A \times A$ and $\langle g_1 m_2, g_2 m_2 \rangle : C_2 \to A \times A$, and factorise it as an epimorphism $[h_1, h_2]\colon A + C_2 \to R$ followed by a monomorphism $\langle r_1, r_2 \rangle : R \to A \times A$. We shall show that R is A_2, with $h_1 = d$ and $h_2 = m_2$, so that $[d, m_2]$ is an epimorphism, as required.

If we regard R as a relation on A, it is clearly reflexive, since by construction it contains the diagonal; it is symmetric, since the relations A and C_2 are so. The pullback $(A + C_2) \times_A (A + C_2)$ is given by $A + C_2 + C_2 + C_3$, where $C_3 = C \times_B C \times_B C$, and the "composition" map $C_3 \to C_2$ sending a triple (c_1, c_2, c_3) of generalized elements of C to (c_1, c_3), induces an evident map $A + C_2 + C_2 + C_3 \to A + C_2$, which in turn induces a map $R \circ R \to R$ showing that the relation R is transitive and so an equivalence relation.

In a topos, an equivalence relation is the kernel pair of its coequaliser (cf Proposition 22), but the coequaliser of $r_1, r_2\colon R \to A$ is the coequaliser of the maps $r_1[h_1, h_2], r_2[h_1, h_2]\colon A + C_2 \to A$, since $[h_1, h_2]$ is epi. This in turn is the coequaliser of the maps $g_1 m_2$ and $g_2 m_2$ and so, using the definition of the g_i, it is the coequaliser of $mf_1, mf_2\colon C_2 \to A$.

$$
\begin{array}{ccc}
C_2 & \xrightarrow{f_2} & C \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\
C & \xrightarrow[f]{} & B
\end{array}
$$

Using the fact that f is epi, the diagram to the right is a pushout (cf Lemma 7). Thus a map $w\colon A \to W$ satisfying $wmf_1 = wmf_2$ induces a unique map $v\colon B \to W$ satisfying $vf = wm$; and so a unique map $u\colon D \to W$ satisfying $ug = w$ and $un = v$. Clearly $g\colon A \to D$ coequalises mf_1, mf_2; using the universal property of coequalisers we obtain that u is an isomorphism. This proves that the coequaliser of the projections of R is $g\colon A \to D$, and so that R is the kernel pair of g; but the kernel pair of g is A_2, and this now proves that $[d, m_2]\colon A + C_2 \to A_2$ is an epimorphism, as claimed. $\square$

We are now able to combine these results in order to deduce our main contribution:

Theorem 26. *Toposes are adhesive.*

Proof. Consider the pushout of Fig 1 in a topos **C**, with m a monomorphism. We shall show that it is a van Kampen square. As a consequence of Proposition 15, all pushouts are stable under pullback.

By parts (iii) and (ii) of Lemma 18, such a pushout is also a pullback and the map n is also a monomorphism. Factorise $g\colon A \to D$ as a epimorphism $q\colon A \to F$ followed by a monomorphism $k\colon F \to D$, and form the pullback squares as illustrated. It follows immediately that j is a monomorphism. Using Lemma 18(i), r is an epimorphism.

$$\begin{array}{ccccc} C & \xrightarrow{r} & E & \xrightarrow{j} & B \\ {\scriptstyle m}\downarrow & (\dagger) & \downarrow{\scriptstyle l} & (\ddagger) & \downarrow{\scriptstyle n} \\ A & \xrightarrow[q]{} & F & \xrightarrow[k]{} & D \end{array}$$

The exterior of the diagram above is a pushout by assumption. We know by Proposition 15 that pushouts are stable under pullback – the stability of this pushout under pullback along k implies that square (†) is also a pushout, and so square (‡) is also a pushout by the usual cancellation properties of pushouts.

If each of these squares is van Kampen then the conclusion of Lemma 2 implies that so is the exterior; thus it will suffice to consider separately square (†) with r and q epimorphisms and m and l monomorphism, and square (‡) with l, n, j and k all monomorphisms. The fact that the latter is van Kampen follows from Corollary 24, while the fact that the former is van Kampen follows from Theorem 25. □

Remark 27. Recall from [11] that the converse of Theorem 26 does not hold. Indeed, adhesive categories are closed under the coslice construction and thus in general are not even extensive.

4 Conclusion

Throughout the paper we have concentrated on the class of adhesive categories which has many examples of interest to computer scientists, in particular those interested in the theory of graph transformation. We have shown that adhesive categories satisfy the special pullback-pushout lemma, which was previously taken as one of the HLR axioms.

Our main result is that toposes are adhesive; the proof relies on exploiting the relationship between van Kampen squares, descent theory [9] and Brown and Janelidze's work [1] on generalised van Kampen theorems. More concretely, we prove that pushouts along monomorphisms in toposes are van Kampen by splitting them into two pushouts and proving that each is van Kampen – the fact that one is van Kampen follows from Brown and Janelidze's well-known theorem and the proof of the other relies on the fact that epimorphisms in toposes are effective for descent.

In future work, we plan to study the ramifications of the fact that toposes are adhesive by using the rewriting theory developed for adhesive categories to study languages with name-passing which are modelled using the Schanuel topos. We also plan to extend our main theorem to show that certain classes of quasitoposes [14] are quasiadhesive. Such a result would not only prove to be of theoretical interest, but would also allow us simple proofs of the quasiadhesivity of many categories of interest to the graph transformation community. This is because it is possible to show that they arise via so called *Artin gluing* [3].

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