Revision of Lecture 4

• Information transferring across channels
  – Channel characteristics and binary symmetric channel
  – Average mutual information

• Average mutual information tells us what happens to information transmitted across channel, or it “characterises” channel
  – But average mutual information is a bit too mathematical (too abstract)
  – As an engineer, one would rather characterises channel by its physical quantities, such as bandwidth, signal power and noise power or SNR

• Also intuitively given source with information rate $R$, one would like to know if channel is capable of “carrying” the amount of information transferred across it
  – In other word, what is the channel capacity?

→ this lecture
Review of Channel Assumptions

• No amplitude or phase distortion by the channel, and the only disturbance is due to additive white Gaussian noise (AWGN), i.e. ideal channel

• In the simplest case, this can be modelled by a binary symmetric channel (BSC)

• The channel error probability $p_e$ of the BSC depends on the noise power $N_P$ relative to the signal power $S_P$, i.e. $\text{SNR} = S_P/N_P$

• Hence $p_e$ could be made arbitrarily small by increasing the signal power

• The channel noise power can be shown to be $N_P = N_0B$, where $N_0/2$ is power spectral density of the noise and $B$ the channel bandwidth

Our aim is to determine the channel capacity $C$, the maximum possible error-free information transmission rate across the channel
Channel Capacity for Discrete Channels

- Shannon's channel capacity $C$ is based on the average mutual information (average conveyed information across the channel), and one possible definition is

$$C = \max \{I(X, Y)\} = \max \{H(Y) - H(Y|X)\} \quad \text{(bits/symbol)}$$

where $H(Y)$ is the average information per symbol at channel output or destination entropy, and $H(Y|X)$ error entropy

- Let $t_i$ be the symbol duration for $X_i$ and $t_{av}$ be the average time for transmission of a symbol, the channel capacity can also be defined as

$$C = \max \{I(X, Y)/t_{av}\} \quad \text{(bits/second)}$$

- $C$ becomes maximum if $H(Y|X) = 0$ (no errors) and the symbols are equiprobable (assuming constant symbol durations $t_i$)

- Channel capacity can be expressed in either (bits/symbol) or (bits/second)
Channel Capacity: Noise-Free Case

• Now $I(X, Y) = H(Y) = H(X)$, but the entropy of the source is given by:

$$H(X) = - \sum_{i=1}^{q} P(X_i) \log_2 P(X_i) \quad \text{(bits/symbol)}$$

• Let $t_i$ be symbol duration for $X_i$; the average time for transmission of a symbol is

$$t_{av} = \sum_{i=1}^{q} P(X_i) \cdot t_i \quad \text{(second/symbol)}$$

• By definition, the channel capacity is

$$C = \max \{ H(X)/t_{av} \} \quad \text{(bits/second)}$$

• Assuming constant symbol durations $t_i = T_s$, the maximum or the capacity is obtained with equiprobable source symbols, $C = \log_2 q/T_s$, and this is the maximum achievable information transmission rate.
Channel Capacity for BSC

- BSC with equiprobable source symbols $P(X_0) = P(X_1) = 0.5$ and variable channel error probability $p_e$ (due to symmetry of BSC, $P(Y_0) = P(Y_1) = 0.5$)

- The channel capacity $C$ (in bits/symbol) is given as

$$C = 1 + (1 - p_e) \log_2(1 - p_e) + p_e \log_2 p_e$$

If $p_e = 0.5$ (worst case), $C = 0$; and if $p_e = 0$ (best case), $C = 1$
Channel Capacity for BSC (Derivation)

\[ P(X_0) = \frac{1}{2} \]

\[ P(Y_0|X_0) = 1 - p_e \]

\[ P(Y_0) = \frac{1}{2} \]

\[ P(Y_0|X_1) = p_e \]

\[ P(X_1) = \frac{1}{2} \]

\[ P(Y_0|X_1) = 1 - p_e \]

\[ P(Y_1) = \frac{1}{2} \]

\[ P(X_0, Y_0) = P(X_0)P(Y_0|X_0) = (1 - p_e)/2, \quad P(X_0, Y_1) = P(X_0)P(Y_1|X_0) = p_e/2 \]

\[ P(X_1, Y_0) = p_e/2, \quad P(X_1, Y_1) = (1 - p_e)/2 \]

\[ I(X, Y) = P(X_0, Y_0) \log_2 \frac{P(Y_0|X_0)}{P(Y_0)} + P(X_0, Y_1) \log_2 \frac{P(Y_1|X_0)}{P(Y_1)} + \]

\[ + P(X_1, Y_0) \log_2 \frac{P(Y_0|X_1)}{P(Y_0)} + P(X_1, Y_1) \log_2 \frac{P(Y_1|X_1)}{P(Y_1)} \]

\[ = \frac{1}{2}(1 - p_e) \log_2 2(1 - p_e) + \frac{1}{2}p_e \log_2 2p_e + \frac{1}{2}p_e \log_2 2p_e + \frac{1}{2}(1 - p_e) \log_2 2(1 - p_e) \]

\[ = 1 + (1 - p_e) \log_2(1 - p_e) + p_e \log_2 p_e \text{ (bits/symbol)} \]
Channel Capacity and Channel Coding

- **Shannon’s theorem**: If information rate $R \leq C$, there exists a coding technique such that information can be transmitted over the channel with arbitrarily small error probability; if $R > C$, error-free transmission is impossible.

- $C$ is the maximum possible error-free information transmission rate.

- Even in noisy channel, there is no obstruction of reliable transmission, but only a limitation of the rate at which transmission can take place.

- Shannon’s theorem does not tell how to construct such a capacity-approaching code.

- Most practical channel coding schemes are far from optimal, but capacity-approaching codes exist, e.g. turbo codes and low-density parity check codes.
Capacity for Continuous Channels

• Entropy of a continuous (analogue) source, where the source output \( x \) is described by the PDF \( p(x) \), is defined by

\[
H(x) = - \int_{-\infty}^{+\infty} p(x) \log_2 p(x) \, dx
\]

• According to Shannon, this entropy attends the maximum for Gaussian PDFs \( p(x) \) (equivalent to equiprobable symbols in the discrete case)

• Gaussian PDF with zero mean and variance \( \sigma_x^2 \):

\[
p(x) = \frac{1}{\sqrt{2\pi \sigma_x}} e^{-(x^2/2\sigma_x^2)}
\]

• The maximum entropy can be shown to be

\[
H_{\text{max}}(x) = \log_2 \sqrt{2\pi e \sigma_x} = \frac{1}{2} \log_2 2\pi e \sigma_x^2
\]
Capacity for Continuous Channels (continue)

- The signal power at the channel input is $S_P = \sigma_x^2$

- Assuming AWGN channel noise independent of the transmitted signal, the received signal power is $\sigma_y^2 = S_P + N_P$, hence

$$H_{\text{max}}(y) = \frac{1}{2} \log_2 2\pi e (S_P + N_P)$$

- Since $I(x, y) = H(y) - H(y|x)$, and $H(y|x) = H(\varepsilon)$ with $\varepsilon$ being AWGN

$$H(y|x) = \frac{1}{2} \log_2 2\pi e N_P$$

- Therefore, the average mutual information

$$I(x, y) = \frac{1}{2} \log_2 \left( 1 + \frac{S_P}{N_P} \right)$$
Shannon-Hartley Law

- With a sampling rate of $f_s = 2 \cdot B$, the analogue channel capacity is given by

$$C = f_s \cdot I(x, y) = B \cdot \log_2 \left( 1 + \frac{S_P}{N_P} \right) \text{ (bits/second)}$$

where $B$ is the signal bandwidth

- For digital communications, $B$ (Hz) is equivalent to the channel bandwidth, and $f_s$ the symbol rate (symbols/second)

- Channel noise power is $N_P = N_0 \cdot B$, where $N_0$ is the power spectral density of the channel AWGN

- Obvious implications:
  - Increasing the SNR $\frac{S_P}{N_P}$ increases the channel capacity
  - Increasing the channel bandwidth $B$ increases the channel capacity
Bandwidth and SNR Trade off

• From the definition of channel capacity, we can trade the channel bandwidth $B$ for the SNR or signal power $S_P$, and vice versa.

• Depending on whether $B$ or $S_P$ is more precious, we can increase one and reduce the other, and yet maintain the same channel capacity.

• A noiseless analogue channel ($S_P/N_P = \infty$) has an infinite capacity.

• $C$ increases as $B$ increases, but it does not go to infinity as $B \to \infty$; rather $C$ approaches an upper limit:

$$C = B \log_2 \left( 1 + \frac{S_P}{N_0 B} \right) = \frac{S_P}{N_0} \log_2 \left( 1 + \frac{S_P}{N_0 B} \right)^{N_0 B / S_P}$$

Recall that

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

We have

$$C_\infty = \lim_{B \to \infty} C = \frac{S_P}{N_0} \log_2 e = 1.44 \frac{S_P}{N_0}$$
Bandwidth and SNR Trade off – Example

• Q: A channel has an SNR of 15. If the channel bandwidth is reduced by half, determine the increase in the signal power required to maintain the same channel capacity

• A:

\[
B \cdot \log_2 \left(1 + \frac{S_P}{N_0B}\right) = B' \cdot \log_2 \left(1 + \frac{S'_P}{N_0B'}\right)
\]

\[
4 \cdot B = \frac{B}{2} \cdot \log_2 \left(1 + \frac{(S'_P/S_P) \cdot S_P}{N_0B/2}\right)
\]

\[
8 = \log_2 \left(1 + 30 \frac{S'_P}{S_P}\right)
\]

\[
256 = 1 + 30 \frac{S'_P}{S_P} \quad \rightarrow \quad S'_P = 8.5S_P
\]
Summary

- Channel capacity for discrete channels
- Channel capacity for continuous channels
- Shannon theorem
- Bandwidth and signal power trade off