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Brief paper

Robust Finite Word Length controller design[☆]

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ABSTRACT

A novel Finite Word Length (FWL) controller design is proposed in the framework of a mixed μ theory. A robust FWL controller performance measure is first developed, which takes into account the standard robust control requirements as well as the FWL implementation considerations, and the corresponding controller design problem is naturally reformulated as a mixed μ problem which can be treated effectively with the results of the mixed μ theory.

1. Introduction

Robust control capable of coping with uncertainty in plant dynamics has been the focal point of the control community for the past three decades. An implicit assumption in most of the existing robust design methods is that controllers are implemented exactly, i.e. there is no uncertainty occurring in realising controllers. In reality controllers are implemented with Finite Word Length (FWL) processors. In 1997, the fragility problem was raised in the work of (Keel & Bhattacharyya, 1997) which showed by examples that a controller achieving the largest robustness to plant uncertainty most likely has a vanishingly small closed-loop stability margin with respect to the controller parameters. Thus, a control system designed by maximising its robustness to plant uncertainty may be fragile, and the resulting fragile controller will need a processor with a very long bit length in implementation to minimise the FWL effects and therefore avoid degrading the designed closed-loop performance or even destabilising the designed stable closed-loop system. However, in many practical systems, such as mass-produced electronic consumer goods, fixed-point processors of short word length are preferred because of their advantages in component cost, chip area, operation simplicity and power consumption. Therefore, it is not a practical approach to simply pursue the optimal robustness to plant uncertainty without considering the FWL effects (Franklin, Powell & Workman, 1998; Gevers & Li, 1993; Istepanian & Whidborne, 2001).

A suitable robust design approach is maintaining a suboptimal robustness to plant uncertainty while simultaneously making the controller tolerance to FWL implementation as large as possible. Through this design, a robust controller can be obtained which does not require a long word-length hardware for implementation. There exist two types of main FWL errors in digital controller implementation. The first one is the rounding errors that occur in arithmetic operations, and the second one is the parameter representation errors. Typically, these two types of errors are investigated separately for the reason of mathematical tractability. In this paper we deal with the second type of FWL errors. Specifically we consider FWL parameter representation errors in the design of robust controllers.

Most of the existing researches (Collins & Zhao, 2001; D'Andrea & Istepanian, 2002; Mahmoud, 2004, 2005; Norlander & Mäkilä, 2001; Park, 2004; Yang, Wang, & Soh, 2000, 2001; Yee, Yang & Wang, 2000; Yee, Yang, & Wang, 2001) refer to robust digital control design with the consideration of FWL parameter representation errors as non-fragile/defragile/resilient control. The works (Mahmoud, 2004, 2005; Park, 2004; Yang et al., 2000;

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Yee et al., 2001) hypothesize that the controller parameter perturbation block is 2-norm bounded, while the works (Collins & Zhao, 2001; D'Andrea & Istepanian, 2002; Norlander & Mäkilä, 2001; Yang et al., 2001; Yee et al., 2000) use a more suitable hypothesis which claims that every parameter perturbation is independent and is magnitude bounded. The methods of D'Andrea and Istepanian (2002) and Yee et al. (2000) deal with static state feedback, while the LQ control of a fixed order and PID control are studied in Norlander and Mäkilä (2001) and Collins and Zhao (2001), respectively. A common Lyapunov function matrix is sought in Yang et al. (2001) for all the vertices of the FWL perturbation hypercube in designing a H_2 controller of fixed order, but the huge number of these vertices can result in excessive complexity in practical computation.

Under the suitable hypothesis similar to the one used in Collins and Zhao (2001), D'Andrea and Istepanian (2002), Norlander and Mäkilä (2001), Yang et al. (2001) and Yee et al. (2000) we study the design of the H_∞ output feedback controller of a fixed order with FWL considerations. A novel FWL robust control performance measure is proposed which takes into account the standard robust control requirements, such as plant uncertainties and input–output characteristics, as well as the FWL effects on controller implementation. We show that the related robust FWL controller design problem can naturally be formulated as a mixed μ problem, and thus it can be solved effectively with the aid of the mixed μ theory. Our proposed robust FWL controller design is also computationally more attractive than the existing design methods, such as the one introduced in Yang et al. (2001) which suffers from high-dimensionality difficulty.

The remainder of this paper is organised in the following way. Notations and preliminaries are offered in Section 2. Section 3 presents a robust FWL performance measure, while Section 4 derives the proposed design approach through optimising this measure. Two numerical examples are given in Section 5 to demonstrate the effectiveness of our proposed method, and the paper concludes at Section 6.

2. Notations and preliminaries

Let $\mathcal{R}$ be the field of real numbers and $\mathcal{C}$ the field of complex numbers, while $\mathcal{U}$ is the closed unit disk in $\mathcal{C}$. For a matrix $\mathbf{A}$, $\mathbf{A} > 0$ means that $\mathbf{A}$ is a positive definite matrix, $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$, and $\mathbf{A}^*$ the complex conjugate transpose of $\mathbf{A}$. The largest singular value of $\mathbf{A}$ is denoted by $\overline{\sigma}(\mathbf{A})$. $\|\mathbf{A}\|_{\mathrm{F}}$ is the Frobenius norm of $\mathbf{A}$, while $\|\mathbf{A}\|_{\mathrm{m}}$ is the modulus of the entry whose modulus is the largest among all the entries of $\mathbf{A}$. $\rho(\mathbf{A})$ and $\det \mathbf{A}$ represent the spectral radius and the determinant of square matrix $\mathbf{A}$, respectively. $\mathbf{I}_n$ is the $n \times n$ identity matrix, while $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. Let $\mathbf{d}_n = [1 \ \cdots \ 1] \in \mathcal{R}^{1\times n}$be the $1 \times n$ row vector whose elements are all equal to 1. $\mathbf{A} \bigotimes \mathbf{B}$ is the Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$.

Denote $\mathcal{F}$ the set of all the causal finite-dimensional linear time-invariant discrete-time systems. Any system in $\mathcal{F}$ can be described as

$$\begin{cases}\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k),\\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k),\end{cases} \tag{1}$$

where the real constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have appropriate dimensions. The transfer function matrix of the above system is

$$\hat{\mathbf{G}}(w) \triangleq w\mathbf{C}(\mathbf{I} - w\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{2}$$

$\hat{\mathbf{G}}(w)$ is stable ($\mathbf{A}$ is stable) if and only if $\rho(\mathbf{A}) < 1$ or equivalently $\forall w \in \mathcal{U}$, $\det(\mathbf{I} - w\mathbf{A}) \neq 0$. The H_∞ norm and H_2 norm of stable $\hat{\mathbf{G}}(w)$ are defined as

$$\|\hat{\mathbf{G}}(w)\|_\infty \triangleq \sup_{w\in\mathcal{U}} \overline{\sigma}(\hat{\mathbf{G}}(w)) < \infty, \tag{3}$$

$$\|\hat{\mathbf{G}}(w)\|_2 \triangleq \left(\|\mathbf{D}\|_{\mathrm{F}}^2 + \sum_{i=0}^{\infty} \|\mathbf{C}\mathbf{A}^i\mathbf{B}\|_{\mathrm{F}}^2\right)^{1/2} < \infty, \tag{4}$$

respectively. For a discrete-time stable system, its H_∞ norm is no less than its H_2 norm.

The following results of the mixed μ theory are from Young (1993). Suppose that we have a matrix $\mathbf{M} \in \mathcal{C}^{n_a\times n_a}$ and three non-negative integers p, q and r with $p + q + r \leq n_a$, which specify the numbers of uncertainty blocks of three types: repeated complex scalars, repeated real scalars and full complex blocks. A $(p+q+r)$-tuple of positive integers

$$\mathbf{k}(p, q, r) = \begin{bmatrix}k_1 & \cdots & k_p & k_{p+1} & \cdots & k_{p+q} & m_1 & \cdots & m_r\end{bmatrix}^{\mathrm{T}} \tag{5}$$

specifies the dimensions of the perturbation blocks, and

$$\sum_{i=1}^{p+q} k_i + \sum_{j=1}^{r} m_j = n_a$$

in order that these dimensions are compatible with $\mathbf{M}$. The block structure $\mathbf{k}(p, q, r)$ determines the set of allowable perturbations, namely,

$$\mathcal{K} \triangleq \left\{ \mathbf{\Upsilon} \left| \begin{array}{l} \mathbf{\Upsilon} = \mathrm{diag}(\zeta_1\mathbf{I}_{k_1}, \cdots, \zeta_p\mathbf{I}_{k_p}, \\ \zeta_{p+1}\mathbf{I}_{k_{p+1}}, \ldots, \zeta_{p+q}\mathbf{I}_{k_{p+q}}, \mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_r) \\ \forall i \in \{1, \ldots, p\}, \zeta_i \in \mathcal{C} \\ \forall i \in \{p+1, \ldots, p+q\}, \zeta_i \in \mathcal{R} \\ \forall j \in \{1, \ldots, r\}, \mathbf{\Gamma}_j \in \mathcal{C}^{m_j\times m_j} \end{array} \right. \right\}.$$

The mixed μ of a matrix $\mathbf{M} \in \mathcal{C}^{n_a\times n_a}$ with respect to a perturbation set $\mathcal{K}$ is defined as

$$\mu_{\mathcal{K}}(\mathbf{M}) \triangleq \left(\inf_{\mathbf{\Upsilon}\in\mathcal{K}} \{\overline{\sigma}(\mathbf{\Upsilon}) | \det(\mathbf{I} - \mathbf{\Upsilon}\mathbf{M}) = 0\}\right)^{-1}. \tag{6}$$

Lemma 1. *Suppose that* $p = 1$, $q = 0$ *and* $r = 0$. *Then* $\mu_{\mathcal{K}}(\mathbf{M}) = \rho(\mathbf{M})$.

Presently, except for a few special cases, how to compute $\mu_{\mathcal{K}}(\mathbf{M})$ is unknown. However, an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$ provided in the following is easy to compute and is often used to replace $\mu_{\mathcal{K}}(\mathbf{M})$ in practice. Define

$$\mathcal{E}_{\mathcal{K}} \triangleq \left\{ \mathbf{E} \left| \begin{array}{l} \mathbf{E} = \mathrm{diag}(\mathbf{E}_1, \cdots, \mathbf{E}_{p+q}, \eta_1\mathbf{I}_{m_1}, \ldots, \eta_r\mathbf{I}_{m_r}) \\ \forall i \in \{1, \ldots, p+q\}, 0 < \mathbf{E}_i \in \mathcal{C}^{k_i\times k_i} \\ \forall j \in \{1, \ldots, r\}, 0 < \eta_j \in \mathcal{R} \end{array} \right. \right\}$$

$$\mathcal{G}_{\mathcal{K}} \triangleq \left\{ \mathbf{G} \left| \begin{array}{l} \mathbf{G} = \mathrm{diag}(0\mathbf{I}_{k_1}, \ldots, 0\mathbf{I}_{k_p}, \\ \mathbf{G}_{p+1}, \ldots, \mathbf{G}_{p+q}, 0\mathbf{I}_{m_1}, \ldots, 0\mathbf{I}_{m_r}) \\ \forall i \in \{p+1, \ldots, p+q\}, \mathbf{G}_i = \mathbf{G}_i^* \in \mathcal{C}^{k_i\times k_i} \end{array} \right. \right\}.$$

Then an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$ is

$$\alpha_{\mathcal{K}}(\mathbf{M}) \triangleq \inf_{\substack{\mathbf{E}\in\mathcal{E}_{\mathcal{K}}\\ \mathbf{G}\in\mathcal{G}_{\mathcal{K}}\\ 0<\alpha\in\mathcal{R}}} \left\{ \alpha \left| \begin{array}{l} \alpha^2\mathbf{E} - \mathbf{M}^*\mathbf{E}\mathbf{M} \\ -\sqrt{-1}(\mathbf{G}\mathbf{M} - \mathbf{M}^*\mathbf{G}) > 0 \end{array} \right. \right\}. \tag{7}$$

When the real scalars of $\mathbf{\Upsilon} \in \mathcal{K}$ are not repeated and $\mathbf{M}$ is a real matrix, $\alpha_{\mathcal{K}}(\mathbf{M})$ can be expressed and computed more simply. Define $\mathcal{E}_{\mathcal{R}\mathcal{K}} \triangleq \left\{\mathbf{E} \in \mathcal{E}_{\mathcal{K}} \left| \mathbf{E} \in \mathcal{R}^{n_a\times n_a}\right.\right\}$.

Lemma 2. *Suppose that we have a real matrix* $\mathbf{M} \in \mathcal{R}^{n_a \times n_a}$ *and a perturbation set* $\mathcal{K}$ *with* $k_i = 1$ *for* $i \in \{p+1, \ldots, p+q\}$ *(i.e. none of the real scalars are repeated). Then*

$$\alpha_{\mathcal{K}}(\mathbf{M}) = \inf_{\substack{\mathbf{E} \in \mathcal{E}_{\mathcal{RK}} \\ 0<\alpha\in\mathcal{R}}} \{\alpha \mid \alpha^2 \mathbf{E} - \mathbf{M}^{\mathrm{T}}\mathbf{E}\mathbf{M} > 0\}. \tag{8}$$

Corollary 1. *For* $\mathbf{M}$ *and* $\mathcal{K}$ *as in Lemma 2,* $\alpha_{\mathcal{K}}(\mathbf{M}) < 1$ *if and only if there exists* $\mathbf{E} \in \mathcal{E}_{\mathcal{RK}}$ *such that* $\mathbf{E} - \mathbf{M}^{\mathrm{T}}\mathbf{E}\mathbf{M} > 0$.

Consider a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ partitioned as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{1,1} & \mathbf{M}_{1,2} \\ \mathbf{M}_{2,1} & \mathbf{M}_{2,2} \end{bmatrix}$$

with square $\mathbf{M}_{1,1}$ and $\mathbf{M}_{2,2}$. The perturbation sets $\mathcal{K}_1$ and $\mathcal{K}_2$ are compatible with $\mathbf{M}_{1,1}$ and $\mathbf{M}_{2,2}$, respectively. Then the perturbation set defined in (9) is compatible with $\mathbf{M}$.

$$\mathcal{K}_f \triangleq \{\boldsymbol{\Upsilon} = \mathrm{diag}(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2) | \boldsymbol{\Upsilon}_1 \in \mathcal{K}_1, \boldsymbol{\Upsilon}_2 \in \mathcal{K}_2\}. \tag{9}$$

Lemma 3. *For* $0 < \alpha \in \mathcal{R}$, $\mu_{\mathcal{K}_f}(\mathbf{M}) < \alpha$ *if and only if* $\mu_{\mathcal{K}_1}(\mathbf{M}_{1,1}) < \alpha$ *and* $\forall \boldsymbol{\Upsilon}_1 \in \mathcal{K}_1$ *with* $\overline{\sigma}(\boldsymbol{\Upsilon}_1) \le \frac{1}{\alpha}$, $\mu_{\mathcal{K}_2}(F(\mathbf{M}, \boldsymbol{\Upsilon}_1)) < \alpha$, *where*

$$F(\mathbf{M}, \boldsymbol{\Upsilon}_1) \triangleq \mathbf{M}_{2,2} + \mathbf{M}_{2,1}(\mathbf{I} - \boldsymbol{\Upsilon}_1\mathbf{M}_{1,1})^{-1}\boldsymbol{\Upsilon}_1\mathbf{M}_{1,2}. \tag{10}$$

3. FWL robust performance measure $\widetilde{\upsilon}$

The plant is described by a known nominal model $\hat{\mathbf{P}}_g(w)$ and an unknown but bounded structured uncertainty $\hat{\mathbf{U}}(w)$. The model $\hat{\mathbf{P}}_g(w)$ is given as

$$\begin{cases} \mathbf{x}_P(k+1) = \mathbf{A}_P\mathbf{x}_P(k) + \mathbf{B}_v\mathbf{v}(k) + \mathbf{B}_w\mathbf{w}(k) + \mathbf{B}_P\mathbf{u}_P(k), \\ \mathbf{h}(k) = \mathbf{C}_h\mathbf{x}_P(k) + \mathbf{D}_{1,1}\mathbf{v}(k) + \mathbf{D}_{1,2}\mathbf{w}(k), \\ \mathbf{z}(k) = \mathbf{C}_z\mathbf{x}_P(k) + \mathbf{D}_{2,1}\mathbf{v}(k) + \mathbf{D}_{2,2}\mathbf{w}(k) + \mathbf{D}_{2,3}\mathbf{u}_P(k), \\ \mathbf{y}_P(k) = \mathbf{C}_P\mathbf{x}_P(k) + \mathbf{D}_{3,2}\mathbf{w}(k), \end{cases} \tag{11}$$

where state $\mathbf{x}_P(k) \in \mathcal{R}^n$, uncertainty-linked input $\mathbf{v}(k) \in \mathcal{R}^{n_1}$, external disturbance input $\mathbf{w}(k) \in \mathcal{R}^{n_2}$, control input $\mathbf{u}_P(k) \in \mathcal{R}^s$, uncertainty-linked output $\mathbf{h}(k) \in \mathcal{R}^{n_1}$, controlled output $\mathbf{z}(k) \in \mathcal{R}^{n_2}$, and measured output $\mathbf{y}_P(k) \in \mathcal{R}^t$. Note that we have assumed without loss of generality that $\mathbf{v}(k)$ and $\mathbf{h}(k)$ have the same dimension as well as that $\mathbf{w}(k)$ and $\mathbf{z}(k)$ have the same dimension. If the dimensions of the paired two variables are different, they can always be made equal by adding an appropriate number of zero rows/columns to the corresponding plant matrices. In addition, it is assumed that $\mathbf{B}_P^{\mathrm{T}}\mathbf{B}_P > 0$ and $\mathbf{C}_P\mathbf{C}_P^{\mathrm{T}} > 0$. This assumption reflects a reasonable practical situation of no redundant actuator or sensor.

Through $\mathbf{h}$ and $\mathbf{v}$, $\hat{\mathbf{P}}_g(w)$ connects with the structured uncertainty $\hat{\mathbf{U}}(w)$, i.e.

$$\mathbf{v} = \hat{\mathbf{U}}(w)\mathbf{h} = \mathrm{diag}\left(\hat{\mathbf{U}}_1(w), \ldots, \hat{\mathbf{U}}_{b+d}(w)\right)\mathbf{h}, \tag{12}$$

where $\hat{\mathbf{U}}_i(w) = \varphi_i(w)\mathbf{I}_{p_i}$ with $\varphi_i(w) \in \mathcal{C}, \forall w \in \mathcal{C}, \forall i \in \{1, \ldots, b\}$, and $\hat{\mathbf{U}}_i(w) \in \mathcal{C}^{p_i \times p_i}, \forall w \in \mathcal{C}, \forall i \in \{b+1, \ldots, b+d\}$, while

$$\sum_{i=1}^{b+d} p_i = n_1, \quad p_i \ge 1.$$

It is assumed that the above $\hat{\mathbf{U}}(w)$ is included in the set

$$\mathcal{H}_\tau \triangleq \left\{ \hat{\mathbf{U}}(w) \left| \begin{array}{l} \hat{\mathbf{U}}(w) = \mathrm{diag}\left(\hat{\mathbf{U}}_1(w), \ldots, \hat{\mathbf{U}}_{b+d}(w)\right) \\ \hat{\mathbf{U}}(w) \in \mathcal{F}, \hat{\mathbf{U}}(w) \text{is stable}, \\ \|\hat{\mathbf{U}}(w)\|_\infty < \tau \end{array} \right. \right\}$$

with a given constant $\tau > 0$.

The digital controller $\hat{\mathbf{C}}(w)$ of the mth-order is described by

$$\begin{cases} \mathbf{x}_C(k+1) = \mathbf{A}_C\mathbf{x}_C(k) + \mathbf{B}_C\mathbf{y}_P(k) \\ \mathbf{u}_P(k) = \mathbf{C}_C\mathbf{x}_C(k) + \mathbf{D}_C\mathbf{y}_P(k) \end{cases} \tag{13}$$

with $\mathbf{A}_C \in \mathcal{R}^{m \times m}$, $\mathbf{B}_C \in \mathcal{R}^{m \times t}$, $\mathbf{C}_C \in \mathcal{R}^{s \times m}$ and $\mathbf{D}_C \in \mathcal{R}^{s \times t}$. Let us denote

$$\mathbf{X} \triangleq \begin{bmatrix} \mathbf{D}_C & \mathbf{C}_C \\ \mathbf{B}_C & \mathbf{A}_C \end{bmatrix} \in \mathcal{R}^{(s+m)\times(t+m)}.$$

When $\mathbf{X}$ is implemented in a fixed-point format of FWL, it is perturbed into $\mathbf{X} + \boldsymbol{\Delta}$ with $\boldsymbol{\Delta}$ belonging to the hypercube

$$\mathcal{D}_\beta \triangleq \{\boldsymbol{\Delta} \mid \boldsymbol{\Delta} \in \mathcal{R}^{(s+m)\times(t+m)}, \|\boldsymbol{\Delta}\|_{\mathrm{m}} \le \beta\}, \tag{14}$$

where $0 \le \beta \in \mathcal{R}$ is the maximum representation error of the fixed-point digital processor. Denote

$$N \triangleq (s+m)(t+m), \tag{15}$$

$$\mathcal{O} \triangleq \{\mathbf{Q} \mid \mathbf{Q} \in \mathcal{R}^{N\times N}, \mathbf{Q} \text{ is diagonal}\}, \tag{16}$$

$$\mathcal{O}_\beta \triangleq \{\mathbf{Q} \mid \mathbf{Q} \in \mathcal{O}, \overline{\sigma}(\mathbf{Q}) \le \beta\}. \tag{17}$$

Further express $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta} \triangleq \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,t+m} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,t+m} \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{s+m,1} & \delta_{s+m,2} & \cdots & \delta_{s+m,t+m} \end{bmatrix}. \tag{18}$$

It is easy to check that

$$\mathbf{X} + \boldsymbol{\Delta} = \mathbf{X} + \left(\mathbf{d}_{t+m} \bigotimes \mathbf{I}_{s+m}\right)\boldsymbol{\Lambda}\left(\mathbf{I}_{t+m} \bigotimes \mathbf{d}_{s+m}^{\mathrm{T}}\right), \tag{19}$$

$$\begin{aligned} \boldsymbol{\Lambda} \triangleq\ & \mathrm{diag}\big(\delta_{1,1}, \delta_{2,1}, \ldots, \delta_{s+m,1}, \delta_{1,2}, \ldots, \delta_{s+m,2}, \\ & \ldots, \delta_{1,t+m}, \ldots, \delta_{s+m,t+m}\big) \in \mathcal{O}_\beta. \end{aligned} \tag{20}$$

The above description represents a closed-loop system consisting of $\hat{\mathbf{P}}_g(w)$ and $\hat{\mathbf{U}}(w)$ as well as $\mathbf{X}$ and $\boldsymbol{\Lambda}$. Denote this closed-loop system as $\hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})$ and the closed-loop transfer function from $\mathbf{w}(k)$ to $\mathbf{z}(k)$ as $\hat{\boldsymbol{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})$. For $0 < \xi \in \mathcal{R}$, a set is defined which consists of all the mth-order robust controllers without FWL consideration, that is,

$$\mathcal{X}_m \triangleq \left\{ \mathbf{X} \left| \begin{array}{l} \mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}, \forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau, \\ \hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{0}) \text{ is stable}, \\ \|\hat{\boldsymbol{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \mathbf{0})\|_\infty \le \xi \end{array} \right. \right\}. \tag{21}$$

To take into account the FWL error $\boldsymbol{\Lambda}$, we propose the following FWL performance measure for $\mathbf{X} \in \mathcal{X}_m$

$$\upsilon(\mathbf{X}) \triangleq \sup_{0\le\beta\in\mathcal{R}} \left\{ \beta \left| \begin{array}{l} \forall \hat{\mathbf{U}}(w) \in \mathcal{H}_\tau, \forall \boldsymbol{\Lambda} \in \mathcal{O}_\beta, \\ \hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda}) \text{ is stable}, \\ \|\hat{\boldsymbol{\Phi}}_{wz}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})\|_\infty \le \xi \end{array} \right. \right\}.$$

For a given $\mathbf{X} \in \mathcal{X}_m$, how to compute the value of $\upsilon(\mathbf{X})$ is unknown. Therefore, a tractable lower bound of $\upsilon(\mathbf{X})$ is derived with the aid of mixed μ. We begin the derivation by "pulling out" $\hat{\mathbf{U}}(w)$ from $\hat{\boldsymbol{\Phi}}(w, \hat{\mathbf{U}}(w), \mathbf{X}, \boldsymbol{\Lambda})$ and considering the composite system of $\hat{\mathbf{P}}_g(w)$, $\mathbf{X}$ and $\boldsymbol{\Lambda}$. The description of this composite system can be obtained as

$$\begin{cases} \mathbf{x}_{PC}(k+1) = \left(\overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u\boldsymbol{\Lambda}\mathbf{C}_u\right)\mathbf{x}_{PC}(k) + \mathbf{B}_{\overline{v}}\mathbf{v}(k) \\ \quad + \overline{\mathbf{B}}(\mathbf{X})\mathbf{w}(k), \\ \mathbf{h}(k) = \mathbf{C}_{\overline{h}}\,\mathbf{x}_{PC}(k) + \mathbf{D}_{1,1}\mathbf{v}(k) + \mathbf{D}_{1,2}\mathbf{w}(k), \\ \mathbf{z}(k) = \overline{\mathbf{C}}(\mathbf{X})\mathbf{x}_{PC}(k) + \mathbf{D}_{2,1}\mathbf{v}(k) + \overline{\mathbf{D}}(\mathbf{X})\mathbf{w}(k), \end{cases} \tag{22}$$

where

$$\bar{\mathbf{A}}(\mathbf{X}) = \begin{bmatrix}\mathbf{A}_P & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{B}_P & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_m\end{bmatrix}\mathbf{X}\begin{bmatrix}\mathbf{C}_P & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_m\end{bmatrix} \triangleq \mathbf{M}_0+\mathbf{M}_1\mathbf{X}\mathbf{M}_2 \in \mathcal{R}^{(n+m)\times(n+m)}, \tag{23}$$

$$\mathbf{B}_u \triangleq \mathbf{d}_{t+m} \bigotimes \mathbf{M}_1 \in \mathcal{R}^{(n+m)\times N}, \tag{24}$$

$$\mathbf{C}_u \triangleq \mathbf{M}_2 \bigotimes \mathbf{d}_{s+m}^{\mathrm{T}} \in \mathcal{R}^{N\times(n+m)}, \tag{25}$$

$$\mathbf{B}_{\overline{v}} = \begin{bmatrix}\mathbf{B}_v^{\mathrm{T}} & \mathbf{0}\end{bmatrix}^{\mathrm{T}} \in \mathcal{R}^{(n+m)\times n_1}, \tag{26}$$

$$\bar{\mathbf{B}}(\mathbf{X}) = \begin{bmatrix}\mathbf{B}_w^{\mathrm{T}} & \mathbf{0}\end{bmatrix}^{\mathrm{T}} + \mathbf{M}_1\mathbf{X}\begin{bmatrix}\mathbf{D}_{3,2}^{\mathrm{T}} & \mathbf{0}\end{bmatrix}^{\mathrm{T}} \triangleq \mathbf{B}_{\overline{w}} + \mathbf{M}_1\mathbf{X}\mathbf{N}_2 \in \mathcal{R}^{(n+m)\times n_2}, \tag{27}$$

$$\mathbf{C}_{\overline{h}} = \begin{bmatrix}\mathbf{C}_h & \mathbf{0}\end{bmatrix} \in \mathcal{R}^{n_1\times(n+m)}, \tag{28}$$

$$\bar{\mathbf{C}}(\mathbf{X}) = \begin{bmatrix}\mathbf{C}_z & \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{D}_{2,3} & \mathbf{0}\end{bmatrix}\mathbf{X}\mathbf{M}_2 \triangleq \mathbf{C}_{\overline{z}} + \mathbf{N}_1\mathbf{X}\mathbf{M}_2 \in \mathcal{R}^{n_2\times(n+m)}, \tag{29}$$

$$\bar{\mathbf{D}}(\mathbf{X}) = \mathbf{D}_{2,2} + \mathbf{N}_1\mathbf{X}\mathbf{N}_2 \in \mathcal{R}^{n_2\times n_2}, \tag{30}$$

$$\mathbf{x}_{PC}(k) \triangleq \begin{bmatrix}\mathbf{x}_P^{\mathrm{T}}(k) & \mathbf{x}_C^{\mathrm{T}}(k)\end{bmatrix}^{\mathrm{T}}. \tag{31}$$

The transfer function matrix of (22) is

$$\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda}) \triangleq w\begin{bmatrix}\mathbf{C}_{\overline{h}}\\ \bar{\mathbf{C}}(\mathbf{X})\end{bmatrix}(\mathbf{I}-w(\bar{\mathbf{A}}(\mathbf{X})+\mathbf{B}_u\mathbf{\Lambda}\mathbf{C}_u))^{-1} \times\begin{bmatrix}\mathbf{B}_{\overline{v}} & \bar{\mathbf{B}}(\mathbf{X})\end{bmatrix} + \begin{bmatrix}\mathbf{D}_{1,1} & \mathbf{D}_{1,2}\\ \mathbf{D}_{2,1} & \bar{\mathbf{D}}(\mathbf{X})\end{bmatrix}, \tag{32}$$

where $\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda}) \in \mathcal{C}^{(n_1+n_2)\times(n_1+n_2)}$ for any $w \in \mathcal{U}$. Let

$$\mathcal{K}_\psi \triangleq \left\{\mathbf{\Upsilon}_\psi \left| \begin{array}{l}\mathbf{\Upsilon}_\psi = \mathrm{diag}(\theta_1\mathbf{I}_{p_1},\ldots,\theta_b\mathbf{I}_{p_b},\\ \mathbf{\Omega}_1,\ldots,\mathbf{\Omega}_{d+1}) \in \mathcal{C}^{(n_1+n_2)\times(n_1+n_2)}\\ \forall i \in \{1,\ldots,b\},\ \theta_i \in \mathcal{C}\\ \forall j \in \{1,\ldots,d\},\ \mathbf{\Omega}_j \in \mathcal{C}^{p_{b+j}\times p_{b+j}}\\ \mathbf{\Omega}_{d+1} \in \mathcal{C}^{n_2\times n_2}\end{array}\right.\right\}.$$

Then, we can obtain the corresponding $\mu_{\mathcal{K}_\psi}(\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda}))$. The following result on robust performance (Zhou, Doyle, & Glover, 1996) links $\upsilon(\mathbf{X})$ to $\mu_{\mathcal{K}_\psi}(\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda}))$.

Lemma 4. *For $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$, if there exists $0 \le \beta \in \mathcal{R}$ such that*

$$\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda}) \text{ is stable}, \forall \mathbf{\Lambda} \in \mathcal{O}_\beta, \tag{33}$$

$$\begin{cases}\mu_{\mathcal{K}_\psi}\left(\mathrm{diag}\left(\tau\mathbf{I}_{n_1}, \dfrac{1}{\xi}\mathbf{I}_{n_2}\right)\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda})\right) < 1,\\ \forall w \in \mathcal{U},\ \forall \mathbf{\Lambda} \in \mathcal{O}_\beta,\end{cases} \tag{34}$$

then $\mathbf{X} \in \mathcal{X}_m$ and $\beta < \upsilon(\mathbf{X})$.

The problem in dealing with (33) and (34) is that $\hat{\mathbf{\Psi}}(w,\mathbf{X},\mathbf{\Lambda})$ contains an indeterminate w and $\mathbf{\Lambda}$. For this reason, we need the following theorem.

Theorem 1. *For $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$, if there exists $0 \le \beta \in \mathcal{R}$ such that*

$$\mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta)) < 1, \tag{35}$$

then (33) and (34) hold. In (35),

$$\mathbf{\Theta}(\mathbf{X},\beta) \triangleq \begin{bmatrix}\bar{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u & \mathbf{B}_{\overline{v}} & \bar{\mathbf{B}}(\mathbf{X})\\ \beta\mathbf{C}_u & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \tau\mathbf{C}_{\overline{h}} & \mathbf{0} & \tau\mathbf{D}_{1,1} & \tau\mathbf{D}_{1,2}\\ \dfrac{1}{\xi}\bar{\mathbf{C}}(\mathbf{X}) & \mathbf{0} & \dfrac{1}{\xi}\mathbf{D}_{2,1} & \dfrac{1}{\xi}\bar{\mathbf{D}}(\mathbf{X})\end{bmatrix}, \tag{36}$$

$$\mathcal{K}_\theta \triangleq \left\{\mathrm{diag}(\mathbf{\Upsilon}_h,\mathbf{\Upsilon}_\psi) | \mathbf{\Upsilon}_h \in \mathcal{K}_h, \mathbf{\Upsilon}_\psi \in \mathcal{K}_\psi\right\}, \tag{37}$$

$$\mathcal{K}_h \triangleq \{\mathbf{\Upsilon}_h = \mathrm{diag}(w\mathbf{I}_{n+m},\mathbf{\Lambda}) | w \in \mathcal{C}, \mathbf{\Lambda} \in \mathcal{O}\}. \tag{38}$$

Proof. Denote

$$\mathbf{H}(\mathbf{X},\beta) \triangleq \begin{bmatrix}\bar{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u\\ \beta\mathbf{C}_u & 0\mathbf{I}_N\end{bmatrix}.$$

By Lemma 3, (35) is equivalent to

$$\mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta)) < 1, \tag{39}$$

$$\mu_{\mathcal{K}_\psi}(F(\mathbf{\Theta}(\mathbf{X},\beta),\mathbf{\Upsilon}_h)) < 1, \quad \forall \mathbf{\Upsilon}_h \in \mathcal{B}\mathcal{K}_h, \tag{40}$$

where $\mathcal{B}\mathcal{K}_h \triangleq \{\mathbf{\Upsilon}_h \mid \mathbf{\Upsilon}_h \in \mathcal{K}_h, \overline{\sigma}(\mathbf{\Upsilon}_h) \le 1\}$. Define

$$\mathcal{K}_a \triangleq \{w\mathbf{I}_{n+m} \mid w \in \mathcal{C}\}, \qquad \mathcal{K}_0 \triangleq \mathcal{O} \tag{41}$$

which are compatible with $\bar{\mathbf{A}}(\mathbf{X})$ and $0\mathbf{I}_N$, respectively. Since $\bar{\mathbf{A}}(\mathbf{X})$ is stable and $\mathcal{K}_a$ contains perturbations of one repeated complex scalar, we conclude by Lemma 1 that

$$\mu_{\mathcal{K}_a}(\bar{\mathbf{A}}(\mathbf{X})) = \rho(\bar{\mathbf{A}}(\mathbf{X})) < 1. \tag{42}$$

Thus, again from Lemma 3, (39) means that $\forall w \in \mathcal{U}$,

$$\mu_{\mathcal{K}_0}(F(\mathbf{H}(\mathbf{X},\beta), w\mathbf{I}_{n+m})) = \mu_{\mathcal{K}_0}(\beta\mathbf{C}_u(\mathbf{I}-w\bar{\mathbf{A}}(\mathbf{X}))^{-1}w\mathbf{B}_u) < 1. \tag{43}$$

It is known from the stability of $\bar{\mathbf{A}}(\mathbf{X})$ that $\mathbf{I}-w\bar{\mathbf{A}}(\mathbf{X})$ is invertible for any $w \in \mathcal{U}$. Then, (43) and (6) imply

$$\begin{aligned}&\inf_{\substack{\mathbf{\Lambda}\in\mathcal{O}\\ w\in\mathcal{U}}}\{\overline{\sigma}(\mathbf{\Lambda}) \mid \det(\mathbf{I}-w(\mathbf{I}-w\bar{\mathbf{A}}(\mathbf{X}))^{-1}\mathbf{B}_u\mathbf{\Lambda}\mathbf{C}_u) = 0\}\\ &= \inf_{\substack{\mathbf{\Lambda}\in\mathcal{O}\\ w\in\mathcal{U}}}\{\overline{\sigma}(\mathbf{\Lambda}) \mid \det(\mathbf{I}-w\bar{\mathbf{A}}(\mathbf{X})-w\mathbf{B}_u\mathbf{\Lambda}\mathbf{C}_u) = 0\}\\ &= \inf_{\mathbf{\Lambda}\in\mathcal{O}}\{\overline{\sigma}(\mathbf{\Lambda}) \mid \bar{\mathbf{A}}(\mathbf{X})+\mathbf{B}_u\mathbf{\Lambda}\mathbf{C}_u \text{ is unstable}\} > \beta.\end{aligned}$$

Thus, (33) holds. For any $\mathbf{\Upsilon}_h = \mathrm{diag}(w\mathbf{I}_{n+m},\mathbf{\Lambda}_0) \in \mathcal{B}\mathcal{K}_h$,

$$\begin{aligned}F(\mathbf{\Theta}(\mathbf{X},\beta),\mathbf{\Upsilon}_h) &= \begin{bmatrix}\tau\mathbf{D}_{1,1} & \tau\mathbf{D}_{1,2}\\ \dfrac{1}{\xi}\mathbf{D}_{2,1} & \dfrac{1}{\xi}\bar{\mathbf{D}}(\mathbf{X})\end{bmatrix} + \begin{bmatrix}\tau\mathbf{C}_{\overline{h}} & \mathbf{0}\\ \dfrac{1}{\xi}\bar{\mathbf{C}}(\mathbf{X}) & \mathbf{0}\end{bmatrix}\\ &\quad\times(\mathbf{I}-\mathbf{\Upsilon}_h\mathbf{H}(\mathbf{X},\beta))^{-1}\mathbf{\Upsilon}_h\begin{bmatrix}\mathbf{B}_{\overline{v}} & \bar{\mathbf{B}}(\mathbf{X})\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\\ &= \begin{bmatrix}\tau\mathbf{D}_{1,1} & \tau\mathbf{D}_{1,2}\\ \dfrac{1}{\xi}\mathbf{D}_{2,1} & \dfrac{1}{\xi}\bar{\mathbf{D}}(\mathbf{X})\end{bmatrix} + w\begin{bmatrix}\tau\mathbf{C}_{\overline{h}}\\ \dfrac{1}{\xi}\bar{\mathbf{C}}(\mathbf{X})\end{bmatrix}\\ &\quad\times(\mathbf{I}-w(\bar{\mathbf{A}}(\mathbf{X})+\beta\mathbf{B}_u\mathbf{\Lambda}_0\mathbf{C}_u))^{-1}\begin{bmatrix}\mathbf{B}_{\overline{v}} & \bar{\mathbf{B}}(\mathbf{X})\end{bmatrix}\\ &= \mathrm{diag}\left(\tau\mathbf{I}_{n_1}, \dfrac{1}{\xi}\mathbf{I}_{n_2}\right)\hat{\mathbf{\Psi}}(w,\mathbf{X},\beta\mathbf{\Lambda}_0).\end{aligned} \tag{44}$$

Thus, (40) guarantees that (34) holds. □

Due to the well-known difficulty in computing the value of $\mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta))$, we replace $\mu_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta))$ with $\alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta))$.

Corollary 2. *For $\mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}$, if there exists $0 \le \beta \in \mathcal{R}$ such that $\alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta)) < 1$, then $\mathbf{X} \in \mathcal{X}_m$ and $\beta < \upsilon(\mathbf{X})$.*

Based on Corollary 2, define

$$\widetilde{\mathcal{X}}_m \triangleq \{\mathbf{X} \mid \mathbf{X} \in \mathcal{R}^{(s+m)\times(t+m)}, \alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},0)) < 1\}, \tag{45}$$

which obviously is a subset of $\mathcal{X}_m$. For $\mathbf{X} \in \widetilde{\mathcal{X}}_m$, define

$$\widetilde{\upsilon}(\mathbf{X}) \triangleq \sup_{0\le\beta\in\mathcal{R}}\{\beta \mid \alpha_{\mathcal{K}_\theta}(\mathbf{\Theta}(\mathbf{X},\beta)) < 1\}, \tag{46}$$